### 6.4.1 Multistep Experiments

The counting rule for multistep experiments, helps us to determine the number of experimental outcomes without listing them. The rule is defined as:

*If an experiment is performed in k stages with $n_1$ ways to accomplish the first stage, $n_2$ ways to accomplish the second stage ... and $n_k$ ways to accomplish the kth stage, then the number of ways to accomplish the experiment is $n_1 \times n_2 \times \ldots \times n_k$.*

**Illustrations**

1. Tossing of two coins can be thought of as a two-step experiment in which each coin can land in one of two ways: head (H) and tail (T). Since the experiment involves two steps, forming the pair of faces (H or T), the total number of simple events in S will be

$$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

The elements of S indicate that there are $2 \times 2 = 4$ possible outcomes.

When the number of alternative events in each of the several trials is same, that is, $n_1 = n_2 = \ldots = n_k$, then the multi-step method gives $n_1 \times n_2 \times \ldots \times n_k = n^k$.
For example, if the coins involved in a coin-tossing experiment are four, then the number of experimental outcomes will be $2 \times 2 \times 2 \times 2 = 2^4 = 16$.

2. Suppose a person can take three routes from city A to city B, four from city B to city C and three from city C to city D. Then the possible routes for reaching from city A to D, while he must travel from A to B to C to D are : (A to B) × (B to C) × (C to D) = $3 \times 4 \times 3 = 36$ ways.

### 6.4.2 Combinations

Sometimes the ordering or arrangement of objects is not important, but only the objects that are chosen. For example, (i) you may not care in what order the books are placed on the shelf, but only which books you are able to shelve. (ii) When a five-person committee is chosen from a group of 10 students, the order of choice is not important because all 5 students will be equal members of committe.

This counting rule for combinations allows us to select $r$ (say) number of outcomes from a collection of $n$ distinct outcomes without caring in what order they are arranged. This rule is denoted by

$$C(n, r) = {}^nC_r = \frac{n!}{r!(n-r)!}$$

where $n! = n(n-1)(n-2)\ldots 3 . 2 . 1$ and $0! = 1$.

The notation ! means *factorial*, for example, $4! = 4 \times 3 \times 2 \times 1 = 24$.

**Important Results**

1. ${}^nC_r = {}^nC_{n-r}$ and ${}^nC_n = 1$.
2. If $n$ objects consist of all $n_1$ of one type, all $n_2$ of another type, and so on upto $n_k$ of the $k$th type, then the total number of selections that can be made of 1, 2, 3 upto $n$ objects is $(n_1 + 1)(n_2 + 1)\ldots(n_k + 1) - 1$.
3. The total number of selections from $n$ objects all different is $2^n - 1$.

### 6.4.3 Permutations

This rule of counting involves ordering or permutations. This rule helps us to compute the number of ways in which $n$ distinct objects can be arranged, taking $r$ of them at a time.

The total number of permutations of $n$ objects taken $r$ at a time is given by

$$P(n, r) = {}^nP_r = \frac{n!}{(n-r)!}$$

By permuting each combination of $r$ objects among themselves, we shall obtain all possible permutations of $n$ objects, $r$ at a time. Each combination gives rise to $r!$ permutations, so that $r!\, C(n, r) = P(n, r) = n!/(n-r)!$.

**Example 6.1:** Of ten electric bulbs, three are defective but it is not known which are defective. In how many ways can three bulbs be selected? How many of these selections will include at least one defective bulb?

**Solution:** Three bulbs out of 10 bulbs can be selected in ${}^{10}C_3 = 120$ ways. The number of selections which include exactly one defective bulb will be ${}^{7}C_2 \times {}^{3}C_1 = 63$.

Similarly, the number of selections which include exactly two and three defective bulbs will be ${}^{7}C_1 \times {}^{3}C_2 = 21$ and ${}^{3}C_3 = 1$, respectively. Thus the total number of selections including at least one defective bulb is 63 + 21 + 1 = 85.

**Example 6.2:** A bag contains 6 red and 8 green balls.

(a) If one ball is drawn at random, then what is the probability of the ball being green?

(b) If two balls are drawn at random, then what is the probability that one is red and the other green?

**Solution:** (a) Since the bag contains 6 red and 8 green balls, therefore it contains 6 + 8 = 14 equally likely outcomes, that is, S = $\{r, g\}$. But one ball out of 14 balls can be drawn in ways, that is,

$$ {}^{14}C_1 = \frac{14!}{1!\,(14-1)!} = 14 \text{ ways} $$

Let A be the event of drawing a green ball. Then out of these 8 green balls, one green ball can be drawn in ${}^{8}C_1$ ways:

$$ {}^{8}C_1 = \frac{8!}{1!\,(8-1)!} = 8 $$

Hence, $$ P(A) = \frac{c(A)}{c(S)} = \frac{8}{14} $$

(b) All exhaustive number of cases, $c(S) = {}^{14}C_2 = \dfrac{14!}{2!\,(14-2)!} = 91.$

Also out of 6 red balls, one red ball can be drawn in ${}^{6}C_1$ ways and out of 8 green balls, one green ball can be drawn in ${}^{8}C_1$ ways. Thus, the total number of favourable cases is:

$$ c(B) = {}^{6}C_1 \times {}^{8}C_1 = 6 \times 8 = 48 $$

Thus $$ P(B) = \frac{c(B)}{c(S)} = \frac{48}{91} $$

**Example 6.3:** Tickets are numbered from 1 to 100. They are well shuffled and a ticket is drawn at random. What is the probability that the drawn ticket has

(a) an even number? (*b*) the number 5 or a multiple of 5?

(c) a number which is greater than 75? (*d*) a number which is a square?

**Solution:** Since any of the 100 tickets can be drawn, therefore exhaustive number of cases are $c(S) = 100$.

(a) Let A be the event of getting an even numbered tickets. Then $c(A) = 50$, and hence

$$ P(A) = 50/100 = 1/2 $$

(b) Let B be the event of getting a ticket bearing the number 5 or a multiple of 5, that is,

$$ B = [5, 10, 15, 20, \ldots, 95, 100] $$

which are 20 in number, $c(B) = 20$. Thus $P(B) = 20/100 = 1/5$.

(c) Let C be the event of getting a ticket bearing a number greater than 75, that is,

$$ C = \{76, 77, \ldots, 100\} $$

which are 25 in number, $c(C) = 25$. Thus $P(C) = 25/100 = 1/4$.

(*d*) Let D be the event of getting a ticket bearing a number which is a square, that is,

$$D = \{1, 4, 9, 16, 25, 36, 49, 64, 81, 100\}$$

which are 10 in number, $c(D) = 10$. Thus $P(D) = 10/100 = 1/10$.

## Conceptual Questions 6A

1. (a) Discuss the different schools of thought on the interpretation of probability. How does each school define probability?
   (b) Describe briefly the various schools of thought on probability. Discuss its importance in business decision-making.
   [*HP Univ., MBA, 1998; Delhi Univ., MBA, 1999*]
   (c) Examine critically the different schools of thought on probability.
   [*Delhi Univ., MBA, 1999; Kumaon Univ., 2000*]
2. Explain what you understand by the term probability. Discuss its importance in busines decision-making.
   [*Delhi Univ., MBA, 2002*]
3. (a) Give the classical and statistical definitions of probability and state the relationship, if any, between the two definitions.
   (b) Critically examine the '*a priori*' definition of probability showing clearly the improvement which the empirical version of probability makes over it.
4. Define independent and mutually exclusive events. Can two events be mutually exclusive and independent simultaneously? Support your answer with an example.
   [*Sukhadia Univ., MBA; Delhi Univ., MBA, 1999*]
5. Compare and contrast the three interpretations of probability.
6. Explain the difference between statistically independent and statistically dependent events.
7. Explain the meaning of each of the following terms:
   (a) Random phenomenon (b) Statistical experiment
   (c) Random event (d) Sample space
8. What do you mean by probability? Explain the importance of probability. [*Madras Univ., MA(Eco), MBA, 2003*]
9. State the multiplicative theorem of probability. How is the result modified when the events are independent.
10. Life insurance premiums are higher for older people, but auto insurance premiums are generally higher for younger people. What does this suggest about the risks and probabilities associated with these two areas of insurance business?
11. Distinguish between the two concepts in each of the following pairs:
    (a) Elementary event and compound events
    (b) Mutually exclusive events and overlapping events
    (c) Sample space and sample point
12. (a) Define the terms joint probability, marginal probability, and conditional probability
    (b) By comparing the three kinds of probabilities (joint, conditional, and marginal), explain what information is provided by each
13. Suppose an entire shipment of 1000 items is inspected and 50 items are found to be defective. Assume the defective items are not removed from the shipment before being sent to a retail out let for sale. If you purchase one item from this shipment, what is the probability that it will be one of the defective items?
14. Suppose you are told that the price of a particular stock will increase with a probability of 0.7.
    (a) How is this probability interpreted?
    (b) Assuming the definition of probability in terms of long-run relative frequencies, how would you find the probability that a stock price will increase?

## Self-Practice Problems 6A

6.1 Three unbiased coins are tossed. What is the probability of obtaining:
(a) all heads (b) two heads
(c) one head (d) at least one head
(e) at least two heads, (f) all tails

6.2 A card is drawn from a well-shuffled deck of 52 cards. Find the probability of drawing a card which is neither a heart nor a king.

6.3 In a single throw of two dice, find the probability of getting (a) a total of 11, (b) a total of 8 or 11, and (c) same number on both the dice.

6.4 Five men in a company of 20 are graduates. If 3 men are picked out of the 20 at random, what is the probability that they are all graduates? What is the probability of at least one graduate?

6.5 A bag contains 25 balls numbered 1 through 25. Suppose an odd number is considered a 'success'. Two balls are drawn from the bag with replacement. Find the probability of getting
(a) two successes (b) exactly one success (c) at least one success (d) no success

6.6 A bag contains 5 white and 8 red balls. Two drawings of 3 balls are made such that (a) the balls are replaced before the second trial, and (b) the balls are not replaced before the second trial. Find the probability that the first drawing will give 3 white and the second 3 red balls in each case. [*MD Univ., BCom; GND Univ., MA 1995; Kerala Univ., MCom, 1998*]

**6.7** Three groups of workers contain 3 men and one woman, 2 men and 2 women, and 1 man and 3 women respectively. One worker is selected at random from each group. What is the probability that the group selected consists of 1 man and 2 women?

[*Nagpur Univ., MCom, 1997*]

**6.8** What is the probability that a leap year, selected at random, will contain 53 Sundays?

[*Agra Univ., MCom; Kurukshetra Univ., MCom, 1996; MD Univ., MCom, 1998*]

**6.9** A university has to select an examiner from a list of 50 persons, 20 of them women and 30 men, 10 of them knowing Hindi and 40 not, 15 of them being teachers and the remaining 35 not. What is the probability of the university selecting a Hindi-knowing woman teacher?

[*Jammu Univ., MCom, 1997*]

## Hints and Answers

**6.1** (a) P(all heads) = 1/8 (b) P(two heads) = 3/8
(c) P(one head) = 3/8 (d) P(at least one head) = 7/8
(e) P(at least two heads) = 4/8 = 1/2
(f) P(all tails) = 1/8.

**6.2** $\text{P(neither a heart nor a king)} = \frac{^{36}C_1}{^{52}C_1} = \frac{36}{52}$

**6.3** c(S) = 36; P(total of 11) = 2/36 P(total of 9 or 11) = 7/36

**6.4** $\text{P(all graduate)} = \frac{^5C_3 \times {^{15}C_0}}{^{20}C_3} = \frac{10 \times 1}{1140} = \frac{1}{114}$

$\text{P(no graduate)} = \frac{^{15}C_3 \times {^5C_0}}{^{20}C_2} = \frac{455 \times 1}{1140} = \frac{91}{28}$

$\text{P(at least one graduate)} = 1 - \frac{91}{228} = \frac{137}{228}$

**6.5** (a) $\text{P(two success)} = \frac{13}{25} \times \frac{13}{25} = \frac{169}{625}$

(b) $\text{P(exactly one success)} = \frac{13}{25} \times \frac{12}{25} + \frac{13}{25} \times \frac{12}{25} = \frac{312}{625}$

(c) P(at least one success) = P(exactly one success) + $\text{P(two success)} = \frac{312}{625} + \frac{169}{625} = \frac{481}{625}$

(d) $\text{P(no success)} = \frac{12}{25} \times \frac{12}{25} = \frac{144}{625}$

**6.6** (a) *When balls are replaced:*

Total number of balls in the bag = 5 + 8 = 13.
3 balls can be drawn from 13 in $^{13}C_3$ ways;
3 white balls can be drawn from 5 in $^5C_3$ ways;
3 red balls can be drawn from 8 in $^8C_3$ ways.
The probability of 3 red balls in the second trial

$= \frac{^5C_3}{^{13}C_3} = \frac{5}{143}$

Probability of 3 red balls in the second trial

$= \frac{^4C_2}{^{12}C_2} = \frac{28}{143}$

The probability of the compound event

$\frac{5}{143} \times \frac{28}{143} = \frac{140}{20449} = 0.007$

(b) *When balls are not replaced:*

At the first trial 3 white balls can be drawn in $^5C_3$ ways.

The probability of drawing three white balls at the first trials $= \frac{^5C_3}{^{13}C_3} = \frac{5}{143}$

When the white balls have been drawn and not replaced, the bag contains 2 white and 8 red balls. Therefore, at the second trial 3 balls can be drawn from 10 in $^{10}C_3$ ways and 3 red balls can be drawn from 8 in $^8C_3$ ways.

The probability of 3 red balls in the second trial

$= \frac{^8C_3}{^{10}C_3} = \frac{7}{15}$

The probability of the compound event

$= \frac{5}{142} \times \frac{7}{15} = \frac{7}{429} = 0.016.$

**6.7** There are three possibilities in this case:

(i) Man is selected from the 1st group and women from 2nd and 3rd groups; or
(ii) Man is selected from the 2nd group and women from the 2st and 3rd groups; or
(iii) Man is selected from the 3rd group and women from 1st and 2nd groups.

The probability of selecting a group of 1 man and 2 women is:

$$\left(\frac{3}{4} \times \frac{2}{4} \times \frac{3}{4}\right) + \left(\frac{2}{4} \times \frac{1}{4} \times \frac{3}{4}\right) + \left(\frac{1}{4} \times \frac{1}{4} \times \frac{2}{4}\right)$$

$$= \frac{9}{32} + \frac{3}{32} + \frac{1}{32} = \frac{13}{32}.$$

**6.8** A leap year consists of 366 days, therefore it contains 52 complete weeks and 2 extra days. These 2 days may make the following 7 combinations:

(i) Monday and Tuesday
(ii) Tuesday and Wednesday
(iii) Wednesday and Thursday
(iv) Thursday and Friday
(v) Friday and Saturday

(vi) Saturday and Sunday
(vii) Sunday and Monday

Of these seven equally likely cases only the last two are favourable. Hence the required probability is 2/7.

**6.9** Probability of selecting a woman = 20/50;

Probability of selecting a teacher = 15/50

Probability of selecting a Hindi-knowing candidate is 10/50

Since the events are independent, the probability of the university selecting a Hindi-knowing woman teacher will be: (20/50) × (15/50) × (10/50) = 3/125.

## 6.5 RULES OF PROBABILITY AND ALGEBRA OF EVENTS

In probability we use set theory notations to simplify the presentation of ideas. As discussed earlier in this chapter, the probability of the occurrence of an event A is expressed as:

$$P(A) = \text{probability of event A occurrence}$$

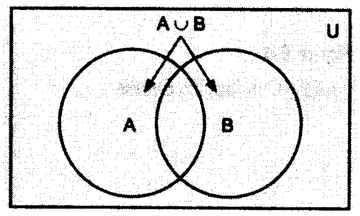
Such single probabilities are called **marginal (or unconditional) probabilities** because it is the probability of a single event occurring. In the coin tossing example, the marginal probability of a tail or head in a toss can be stated as P(T) or P(H).

**Marginal probability:** The unconditional probability of an event occurring.

### 6.5.1 Rules of Addition

The addition rules are helpful when we have two events and are interested in knowing the probability that at least one of the events occurs.

**Figure 6.3**
Union of Two Events




**Mutually Exclusive Events** The rule of addition for mutually exclusive (disjoint), exhaustive, and equally likely events states that

*If two events A and B are mutually exclusive, exhaustive, and equiprobable, then the probability of either event A or B or both occurring is equal to the sum of their individual probabilities.*

This rule is expressed in the following formula

$$P(A \text{ or } B) = P(A \cup B) = \frac{n(A \cup B)}{n(S)} = \frac{n(A) + n(B)}{n(S)}$$

$$= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} = P(A) + P(B) \qquad (6\text{-}1)$$

where $A \cup B$ (read as 'A *union* B') denotes the union of two events A and B and it is the set of all sample points belonging to A or B or both. This rule can also be illustrated by the **Venn diagram** shown in Fig. 6.3. Here two circles contain all the sample points in events A and B. The overlap of the circles indicates that some sample points are contained in both A and B.

**Venn diagram:** A pictorial representation for showing the sample space and operations involving events. The sample space is represented by a rectangle and events as circles.

**Illustration:** Consider the pattern of arrival of customers at a service counter during first hour it is open along with its probability:

| No. of persons : | 0 | 1 | 2 | 3 | 4 or more |
|---|---|---|---|---|---|
| Probability : | 0.1 | 0.2 | 0.3 | 0.3 | 0.1 |

To understand the probability that either 2 or 3 persons will be there during the first hour, we have

$$P(2 \text{ or } 3) = P(2) + P(3) = 0.3 + 0.3 = 0.6$$

The formula (6-1) can be expanded to include more than two events. In particular, if there are $n$ events in a sample space that are mutually exclusive, then the probability of the union of these events is given by

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n) \qquad (6\text{-}2)$$

For example, if we are interested in knowing the probability that there will be two or more persons during the first hour, then using formula (6-2), we have

$$P(2 \text{ or more}) = P(2, 3, 4 \text{ or more}) = P(2) + P(3) + P(4)$$
$$= 0.3 + 0.3 + 0.1 = 0.7$$

An important special case of formula (6-1) is for complementary events. Let A be any event and $\overline{A}$ be the complement of A. Obviously A and $\overline{A}$ are mutually exclusive and exhaustive events. Thus, either A occurs or it does not, is given by

$$P(A \text{ or } \overline{A}) = P(A) + P(\overline{A}) = P(A) + \{1 - P(A)\} = 1$$

or $$P(A) = 1 - P(\overline{A}) \qquad (6\text{-}3)$$

For example, if a dice is rolled, then the probability whether an odd number of spots occurs or does not.

**Partially Overlapping (or Joint) Events** If events A and B are not mutually exclusive, it is possible for both events to occur simultaneously? This means these events have some sample points in common. Such events are also called *joint* (or *overlapping*) *events*. The sample points in common (belong to both events) represent the joint event A ∩ B (read as: A intersection B). The addition rule in this case is stated as:

*If two events A and B are not mutually exclusive, then the probability of either A* or *B or both accuring is equal to the sum of their individual probabilities minus the probability of A and B accuring together.*

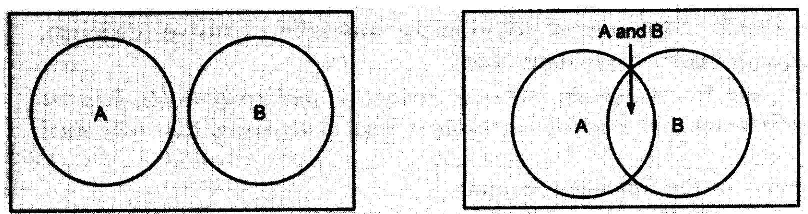
This rule is expressed in the following formula:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

or $$P(A \cup B) = P(A) + P(B) - P(A \cap B) \qquad (6\text{-}4)$$

This addition rule can also be illustrated by the Venn-diagram shown in Fig. 6.4.

**Figure 6.4**
Partially Overlapping Events




(a) Mutually Exclusive Events (b) Partially Overlapping

**Illustration:** Suppose 70 per cent of all tourists who come to India will visit Agra. While 60 per cent will visit Goa and 50 per cent of them will visit both Agra and Goa. The probability that a tourist will visit either Goa or Agra or both is obtained by applying formula (6-4) as follows:

$$P(\text{Agra or Goa}) = P(\text{Agra}) + P(\text{Goa}) - P(\text{both Agra and Goa})$$
$$= 0.70 + 0.60 - 0.50 = 0.8$$

Consequently, the probability that a tourist will visit neither Agra nor Goa is calculated by

$$P(\text{neither Agra nor Goa}) = 1 - P(\text{Agra or Goa}) = 1 - 0.80 = 0.20$$

The formula (6-4) can be expanded to include more than two events. In particular, if there are three events that are not mutually exclusive, then the probability of the union of these events is given by

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C) \qquad (6\text{-}5)$$

**Remark:** The rules of addition are applicable for calculating probability of events in case of simultaneous trails.

**Example 6.4:** What is the probability that a randomly chosen card from a deck of cards will be either a king or a heart.

**Solution:** Let event A and B be the king and heart in a deck of 52 cards, respectively. Then it is given that

| *Card* | *Probability* | *Reason* |
|---|---|---|
| King | P(A) = 4/52 | 4 kings in a deck of 52 cards |
| Heart | P(B) = 13/52 | 13 hearts in a deck of 52 cards |
| King of heart | P(A and (B) = 1/52 | 1 King of heart in a deck of 52 cards |

Using the formula (6-4), we get

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$
$$= \frac{4}{52}+\frac{13}{52}-\frac{1}{52} = \frac{16}{52} = 0.3077$$

**Example 6.5:** Of 1000 assembled components, 10 have a working defect and 20 have a structural defect. There is a good reason to assume that no component has both defects. What is the probability that randomly chosen component will have either type of defect?

*[Delhi Univ., MBA, 2003]*

**Solution:** Let the event A and B be the component which has working defect and has structural defect, respectively. Then it is given that

$$P(A) = 10/1000 = 0.01,\ P(B) = 20/1000 = 0.02 \text{ and } P(A \text{ and } B) = 0$$

The probability that a randomly chosen component will have either type of defect is given by

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$
$$= 0.01 + 0.02 - 0.0 = 0.03$$

**Example 6.6:** A survey of 200 retail grocery shops revealed following monthly income pattern:

| *Monthly Income (Rs)* | *Number of Shops* |
|---|---|
| Under Rs 20,000 | 102 |
| 20,000 to 30,000 | 61 |
| 30,000 and above | 37 |

(a) What is the probability that a particular shop has monthly income under: Rs 20,000

(b) What is the probability that a shop selected at random has either an income between Rs 20,000 and Rs 30,000 or an income of Rs. 30,000 and more?

**Solution:** Let the events A, B and C represent the income under three categories, respectively.

(a) Probability that a particular shop has monthly income under Rs 20,000 is P(A) = 102/200 = 0.51.

(b) Probability that shop selected at random has income between Rs 20,000 and Rs 30,000 or Rs 30,000 and more is given by

$$P(A \text{ or } B) = P(A) + P(B)$$
$$= \frac{61}{200}+\frac{37}{200} = 0.305 + 0.185 = 0.49$$

**Example 6.7:** From a sales force of 150 persons, one will be selected to attend a special sales meeting. If 52 of them are unmarried, 72 are college graduates, and 3/4 of the 52 that are unmarried are college graduates, find the probability that the sales person selected at random will be neither single nor a college graduate.

**Solution:** Let A and B be the events that a sales person selected is married and that he is a college graduate, respectively. Then it is given that

$$P(A) = 52/150,\ P(B) = 72/150;\ P(A \text{ and } B) = (3/4)\ (52/150) = 39/150$$

The probability that a salesperson selected at random will be neither single nor a college graduate is:

$$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B) = 1 - \{P(A) + P(B) - P(A \cap B)\}$$
$$= 1 - \left\{\frac{52}{150} + \frac{72}{150} - \frac{39}{150}\right\} = \frac{13}{30}$$

**Example 6.8:** From a computer tally based on employer records, the personnel manager of a large manufacturing firm finds that 15 per cent of the firm's employees are supervisors and 25 per cent of the firm's employees are college graduates. He also discovers that 5 per cent are both supervisors and college graduates. Suppose an employee is selected at random from the firm's personnel records, what is the probability of:

(a) selecting a person who is both a college graduate and a supervisor?

(b) selecting a person who is neither a supervisor nor a college graduate?

**Solution:** Let A and B be the events that the person selected is a supervisor and that he is a college graduate, respectively. Given that

$$P(A) = 15/100; \quad P(B) = 25/100; \quad P(A \text{ and } B) = 5/100$$

(a) Probability of selecting a person who is both a college graduate and a supervisor is:

$$P(A \text{ and } B) = 5/100 = 0.05$$

(b) Probability of selecting a person who is neither a supervisor nor a college graduate is:

$$P(\bar{A} \text{ and } \bar{B}) = 1 - P(A \text{ or } B) = 1 - [P(A) + P(B) - P(A \text{ and } B)]$$
$$= 1 - \left(\frac{15}{100} + \frac{25}{100} - \frac{5}{100}\right) = \frac{65}{100} = 0.65$$

**Example 6.9:** The probability that a contractor will get a plumbing contract is 2/3 and the probability that he will not get an electrical contract is 5/9. If the probability of getting at least one contract is 4/5, what is the probability that he will get both?

**Solution:** Let A and B denote the event that the contractor will get a plumbing and electrical contract respectively. Given that

$$P(A) = 2/3; \quad P(B) = 1 - (5/9) = 4/9; \quad P(A \cup B) = 4/5$$
$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = \frac{2}{3} + \frac{4}{9} - \frac{4}{5} = \frac{14}{45} = 0.31$$

Thus the probability that the contractor will get both the contracts is 0.31.

**Example 6.10:** An MBA applies for a job in two firms X and Y. The probability of his being selected in firm X is 0.7 and being rejected at Y is 0.5. The probability of at least one of his applications being rejected is 0.6. What is the probability that he will be selected by one of the firms?

**Solution:** Let A and B denote the event that an MBA will be selected in firm X and will be rejected in firm Y, respectively. Then given that

$$P(A) = 0.7, \quad P(\bar{A}) = 1 - 0.7 = 0.3$$
$$P(B) = 0.5, \quad P(\bar{B}) = 1 - 0.5 = 0.5, \quad P(\bar{A} \cup \bar{B}) = 0.6$$

Since $P(A \cap B) = 1 - P(\cup) = 1 - 0.6 = 0.4$

therefore, probability that he will be selected by one of the firms is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 0.7 + 0.5 - 0.4 = 0.8$$

Thus, the probability of an MBA being selected by one of the firms is 0.8.

### 6.5.2 Rules of Multiplication

**Statistically Independent Events** When the occurrence of an event does not affect and is not affected by the probability of occurrence of any other event, the event is said to be a *statistically independent event*. There are three types of probabilities under statistical independence: *marginal, joint,* and *conditional.*

- *Marginal Probability:* A marginal or unconditional probability is the simple probability of the occurrence of an event. For example, in a fair coin toss the outcome of each toss is an event that is statistically independent of the outcomes of every other toss of the coin.
- *Joint Probability:* The probability of two or more independent events occurring together or in succession is called the **joint probability**. The joint probability of two or more independent events is equal to the product of their marginal probabilities. In particular, if A and B are independent events, the probability that both A and B will occur is given by

$$P(AB) = P(A \cap B) = P(A) \times P(B) \qquad (6\text{-}6)$$

Suppose we toss a coin twice. The probability that in both the cases the coin will turn up head is given by

$$P(H_1H_2) = P(H_1) \times P(H_2) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

The formula (6-6) is applied here because the probability of any outcome is not affected by any preceding outcome, in other words, outcomes are independent.

**Joint probability:** The probability of two events occurring together or in succession.

- *Conditional Probability:* It is the probability of a particular event occuring, given that another event has occurred. The **conditional probability** of event A, given that event B has already occurred is written as: P (A|B). Similarly we may write P (B|A). The vertical bar is read as: 'given' and events appearing to the right of the bar are those that you know have occurred. Two events A and B are said to be independent if and only P(A|B) = P(A) or P(B|A) = P(B). Otherwise, events are said to be dependent.

**Conditional probability:** The probability of an event occurring, given that another event has occurred.

Since, in the case of independent events the probability of occurrence of either of the events does not depend or affect the occurrence of the other, therefore in the coin tossing example, the probability of a head occurrence in the second toss, given that head resulted in the first toss, is still 0.5. That is, $P(H_2 \mid H_1) = 0.5 = P(H_2)$. It is because of the fact that the probabilities of heads and tails are the same for every toss and in no way influenced by whether it was a head or tail which occurred in the previous toss.

**Statistical Dependent Events** When the probability of an event is dependent upon or affected by the occurrence of any other event, the events are said to be **statistically dependent**. There are three types of probabilities under statistical dependence: *joint*, *conditional*, and *marginal*.

**Statistical dependence:** The condition when the probability of occurrence of an event is dependent upon, or affected by, the occurrence of some other event.

- *Joint Probability*: If A and B are dependent events, then the joint probability as discussed under statistical independence case is no longer equal to the product of their respective probabilites. That is, for dependent events

$$P(A \text{ and } B) = P(A \cap B) \neq P(A) \times P(B)$$

Accordingly, $P(A) \neq P(A \mid B)$ and $P(B) \neq P(B \mid A)$

The joint probability of events A and B occurring together or in succession under statistical dependences is given by

$$P(A \cap B) = P(A) \times P(B \mid A)$$

or $$P(A \cap B) = P(B) \times P(A \mid B)$$

- *Conditional Probability:* Under statistical dependence, the conditional probability of event B given that event A has already occurred, is given by

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

Similarly the conditional probability of A, given that event B has occurred, is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

- *Marginal Probability*: The marginal probability of an event under statistical dependence is the same as the marginal probability of an event under statistical independence.

The marginal probability of events A and B can be written as:

$$P(A) = P(A \cap B) + P(A \cap \ )$$

and $$P(B) = P(A \cap B) + P(\ \cap B)$$

**Example 6.11:** The odds against student X solving a Business Statistics problem are 8 to 6, and odds in favour of student Y solving the problem are 14 to 16.

(a) What is the chance that the problem will be solved if they both try independently of each other?

(b) What is the probability that none of them is able to solve the problem?

*[Delhi Univ., MBA, 1998]*

**Solution:** Let A = The event that the first student solves the problem,

B = The event that the second student solves the problem.

$$P(A) = \frac{6}{8+6} = \frac{6}{14} \text{ and } P(B) = \frac{14}{14+16} = \frac{14}{30}$$

(a) Probability that the problem will be solved

= P (at least one of them solves the problem)

= P (A or B) = P(A) + P(B) – P(A and B)

= P(A) + P(B) – P(A) × P(B) [because the events are independent]

$$= \frac{6}{14} + \frac{14}{30} - \frac{6}{14} \times \frac{14}{30} = \frac{73}{105} = 0.695$$

(b) Probability that neither A nor B solves the problem

$$P(\bar{A} \text{ and } \bar{B}) = P(\bar{A}) \times P(\bar{B})$$

$$= [1 - P(A)] \times [1 - P(B)] = \frac{8}{14} \times \frac{16}{30} = \frac{32}{105} = 0.305$$

**Example 6.12:** The probability that a new marketing approach will be successful is 0.6. The probability that the expenditure for developing the approach can be kept within the original budget is 0.50. The probability that both of these objectives will be achieved at 0.30.

What is the probability that at least one of these objectives will be achieved. For the two events described above, determine whether the events are independent or dependent.

*[Delhi Univ., MBA, 2003]*

**Solution:** Let A = The event that the new marketing approach will be successful

B = The event that the expenditure for developing the approach can be kept within the original budget

Given that P(A) = 0.60, P(B) = 0.50 and $P(A \cap B) = 0.30$

Probability that both events A and B will be achieved is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.60 + 0.50 - 0.30 = 0.80$$

If events A and B are independent, then their joint probability is given by

$$P(A \cap B) = P(A) \times P(B) = 0.60 \times 0.50 = 0.30$$

Since this value is same as given in the problem, events are independent.

**Example 6.13:** A piece of equipment will function only when the three components A, B, and C are working. The probability of A failing during one year is 0.15, that of B failing is 0.05, and that of C failing is 0.10. What is the probability that the equipment will fail before the end of the year?

**Solution:** Given that

P(A failing) = 0.15; P(A not failing) = 1 – P(A) = 0.85

P(B failing) = 0.05; P(B not failing) = 1 – P(B) = 0.95

P(C failing) = 0.10; P(C not failing) = 1 – P(C) = 0.90

Since all the three events are independent, therefore the probability that the equipment will work is given by

$$P(\bar{A} \cap \bar{B} \cap \bar{B}) = P(\bar{A}) \times P(\bar{B}) \times P(\bar{C}) = 0.85 \times 0.95 \times 0.90 = 0.726$$

Probability that the equipment will fail before the end of the year is given by

$$P(A \cup B \cup C) = 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$$
$$= 1 - P(\bar{A}).\ P(\bar{B}).\ P(\bar{C})$$
$$= 1 - \{0.85 \times 0.95 \times 0.90\} = 1 - 0.726 = 0.274$$

**Example 6.14:** A market research firm is interested in surveying certain attitudes in a small community. There are 125 households broken down according to income, ownership of a telephone, and ownership of a TV.

| | *Households with Annual Income of Rs 8000 or Less* | | *Households with Annual Income Above Rs 8000* | | *Total* |
|---|---|---|---|---|---|
| | *Telephone subscriber* | *No telephone* | *Telephone subscriber* | *No telephone* | |
| Own TV set | 27 | 20 | 18 | 10 | 75 |
| No TV set | 18 | 10 | 12 | 10 | 50 |
| Total | 45 | 30 | 30 | 20 | 125 |

(a) What is the probability of getting a TV owner in at random draw?

(b) If a household has an income of over Rs 8000 and is a telephone subscriber, what is the probability that he owns a TV?

(c) What is the conditional probability of drawing a household that owns a TV, given that the household is a telephone subscriber?

(d) Are the events 'ownership of a TV' and 'telephone subscriber' statistically independent? Comment. [*Himachal Univ., MBA, 1998*]

**Solution:** (a) Probability of drawing a TV owner at random,

P(TV owner) = 75/125 = 0.6

(b) There are 30(18 + 12) persons whose household income is above Rs 8000 and are also telephone subscribers. Out of these, 18 own TV sets. Hence the probability of this group of persons having a TV set, is: 18/30 = 0.6.

(c) Out of 75( 27 + 18 + 18 + 12) households who are telephone subscribers, 45(27 + 18) households have TV sets. Hence the conditional probability of drawing a household that owns a TV given that the household is a telephone subscriber is: 45/75 = 0.6.

(d) Let A and B be the events representing TV owners and telephone subscribers respectively. The probability of a person owning a TV, P(A)= 75/125. The probability of a person being a telephone subscriber, P(B) = 75/125.

The probability of a person being a telephone subscriber as well as a TV owner is:

P(A and B) = 45/125 = 9/25

But P(A) × P(B) = (75/125) (75/125) = 9/25

Since P(AB) = P(A) × P(B), therefore we conclude that the events 'ownership of a TV' and 'telephone subscriber' are statistically independent.

**Example 6.15:** A company has two plants to manufacture scooters. Plant I manufactures 80 per cent of the scooters and Plant II manufactures 20 per cent. In plant I only 85 out of 100 scooters are considered to be of standard quality. In plant II, only 65 out of 100 scooters are considered to be of standard quality. What is the probability that a scooter selected at random came from plant I, if it is known that it is of standard quality?
[*Madras Univ., MCom, 1996; Delhi Univ., MBA, 1998*]

**Solution:** Let A = The scooter purchased is of standard quality

B = The scooter is of standard quality and came from plant I

C = The scooter is of standard quality and came from plant II

D = The scooter came from plant I

The percentage of scooters manufactured in plant I that are of standard quality is 85 per cent of 80 per cent, that is, 0.85 × (80 ÷ 100) = 68 per cent or P(B) = 0.68.

The percentage of scooters manufactured in plant II that are of standard quality is 65 per cent of 20 per cent, that is, 0.65 × (20 ÷ 100) = 13 per cent or P(C) = 0.13.

The probability that a customer obtains a standard quality scooter from the company is, 0.68 + 0.13 = 0.81.

The probability that the scooters selected at random came from plant I, if it is known that it is of standard quality is given by

$$P(D|A) = \frac{P(D \text{ and } A)}{P(A)} = \frac{0.68}{0.81} = 0.84$$

**Example 6.16:** A husband and wife appear in an interview for two vacancies in the same post. The probability of husband's selection is 1/7 and that of wife's selection is 1/5. What is the probability that

(a) both of them will be selected,
(b) only one of them will be selected, and
(c) none of them will be selected.

*[Bharthidasan Univ., MCom, 1996; Delhi Univ., MBA, 1999]*

**Solution:** Let A and B be the event of the husband's and wife's selection, respectively. Given that P(A) = 1/7 and P(B) = 1/5.

(a) The probability that both of them will be selected is:

$$P(A \text{ and } B) = P(A)\,P(B) = (1/7) \times (1/5) = 1/35 = 0.029$$

(b) The probability that only one of them will be selected is:

$$P[(A \text{ and } \bar{B}) \text{ or } (B \text{ and } \bar{A})] = P(A \text{ and } \bar{B}) + P(B \text{ and } \bar{A})$$
$$= P(A)\,P(\bar{B}) + P(B)\,P(\bar{A})$$
$$= P(A)\,[1 - P(B)] + P(B)\,[1 - P(A)]$$
$$= \frac{1}{7}\left(1 - \frac{1}{5}\right) + \frac{1}{5}\left(1 - \frac{1}{7}\right) = \left(\frac{1}{7} \times \frac{4}{5}\right) + \left(\frac{1}{5} \times \frac{6}{7}\right)$$
$$= \frac{10}{35} = 0.286$$

(c) The probability that none of them will be selected is:

$$P(\bar{A}) \times P(\bar{B}) = (6/7) \times (4/5) = 24/35 = 0.686$$

**Example 6.17:** The odds that A speaks the truth is 3 : 2 and the odds that B speaks the truth is 5 : 3. In what percentage of cases are they likely to contradict each other on an identical point? *[Delhi Univ., MBA, 1999]*

**Solution:** Let X and Y denote the events that A and B speaks truth, respectively. Given that

$$P(X) = 3/5;\quad P(\bar{X}) = 2/5;\quad P(Y) = 5/8;\quad P(\bar{Y}) = 3/8$$

The probability that A speaks the truth and B speaks a lie is: (3/5) (3/8) = 9/40

The probability that B speaks the truth and A speaks a lie is: (5/8) (2/5) = 10/40

So the compound probability is: $\frac{9}{40} + \frac{10}{40} = \frac{19}{40}$

Hence, percentage of cases in which they contradict each other is (19/40) × 100 = 47.5 per cent

**Example 6.18:** The data for the promotion and academic qualification of a company is given below:

| *Promotional Status* | *Academic Qualification* *MBA(A)* | *Non-MBA()* | *Total* |
|---|---|---|---|
| Promoted (B) | 0.14 | 0.26 | 0.40 |
| Non-promoted () | 0.21 | 0.39 | 0.60 |
| Total | 0.35 | 0.65 | 1.00 |

(a) Calculate the conditional probability of promotion after an MBA has been identified.
(b) Calculate the conditional probability that it is an MBA when a promoted employee has been chosen.
(c) Find the probability that a promoted employee was an MBA. [*IGNOU, 1995*]

**Solution:** It is given that, P(A)=0.35, P=0.65, P(B)=0.40,P=0.60, and P(A ∩ B)=0.14

(a) The probability of being 'promoted' after an MBA employee has been identified is:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.12}{0.35} = 0.34$$

(b) If a promoted employee has been chosen, then the probability that the person is an MBA is:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.12}{0.40} = 0.30$$

(c) The probability that a promoted employee was an MBA is:

$$P(A \cap B) = P(A) \times P(B \mid A) = 0.35 \times 0.34 = 0.12$$

or

$$P(A \cap B) = P(B) \times P(A \mid B) = 0.40 \times 0.30 = 0.12$$

**Example 6.19:** The probability that a trainee will remain with a company is 0.6. The probability that an employee earns more than Rs 10,000 per month is 0.5. The probability that an employee who is a trainee remained with the company or who earns more than Rs 10,000 per month is 0.7. What is the probability that an employee earns more than Rs 10,000 per month given that he is a trainee who stayed with the company?

**Solution:** Let A and B be the event that a trainee who remained with the company and the event that an employee earns more than Rs 10,000, respectively. Given that

$$P(A) = 0.6,\ P(B) = 0.5,\ \text{and } P(A \text{ or } B) = P(A \cup B) = 0.7$$

The probability that an employee earns more than Rs 10,000 given that he is trainee who remained with the company is given by

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{P(A \cap B)}{P(A)}$$

We know that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$,

or $P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.6 + 0.5 - 0.7 = 0.4$

Hence the required probability is:

$$P(B|A) = \frac{(A \cap B)}{P(A)} = \frac{0.4}{0.6} = 0.667$$

**Example 6.20:** Two computers A and B are to be marketed. A salesman who is assigned the job of finding customers for them has 60 per cent and 40 per cent chances, respectively of succeeding for computers A and B. The two computers can be sold independently. Given that he was able to sell at least one computer, what is the probability that computer A has been sold? [*IGNOU, MBA, 2002; Delhi Univ., MBA, 1999, 2002*]

**Solution:** Let us define the events

$E_1$ = Computer A is marketed and $E_2$ = Computer B is marketed.

It is given that $P(E_1) = 0.60,\ P(E_2) = 0.40$

$$P(E_1 \text{ and } E_2) \text{ or } P(E_1 \cap E_2) = P(E_1).\ P(E_2) = 0.60 \times 0.40 = 0.24$$

Hence, the probability that computer A has been sold given that the salesman was able to sell at least one computer is given by

$$P(E_1 | E_1 \cup E_2) = \frac{P\{E_1 \cap (E_1 \cup E_2)\}}{P(E_1 \cup E_2)} = \frac{P(E_1)}{P(E_1 \cup E_2)}$$

$$= \frac{P(E_1)}{P(E_1) + P(E_2) - P(E_1 \cap E_2)} = \frac{0.60}{0.60 + 0.40 - 0.24}$$

$$= \frac{0.60}{0.76} = 0.789$$

**Example 6.21:** A study of job satisfaction was conducted for four occupations: Cabin maker lawyer, doctor and systems analyst. Job satisfaction was measured on a scale of 0–100. The data obtained are summarized in the following table:

| *Occupation* | *Under 50* | *50–59* | *60–69* | *70–79* | *80–89* | *Total* |
|---|---|---|---|---|---|---|
| Cabin maker | 0 | 2 | 4 | 3 | 1 | 10 |
| Laweyer | 6 | 2 | 1 | 1 | 0 | 10 |
| Doctor | 0 | 5 | 2 | 1 | 2 | 10 |
| Systems Analyst | 2 | 1 | 4 | 3 | 0 | 10 |
| | 8 | 10 | 11 | 8 | 3 | 40 |

(a) Develop a joint probability table.
(b) What is the probability of one of the participants studied had a satisfaction score in 80's?
(c) What is the probability of a satisfaction score in the 80's, given the study participant was a doctor?
(d) What is the probability of one of the participants studied was a lawyer.
(e) What is the probability of one of the participants was a lawyer and received a score under 50?
(f) What is the probability of a satisfaction score under 50 given a person is a lawyer.
(g) What is the probability of a satisfaction score of 70 or higher?

[*Delhi Univ., MBA, 2003*]

**Solution:** (a) Joint probability table is given below

| *Occupation* | *Under 50* | *50–59* | *60–69* | *70–79* | *80–89* |
|---|---|---|---|---|---|
| Cabin maker | 0.000 | 0.050 | 0.100 | 0.075 | 0.250 |
| Lawyer | 0.150 | 0.050 | 0.025 | 0.025 | 0.250 |
| Doctor | 0.000 | 0.125 | 0.050 | 0.025 | 0.250 |
| System Analyst | 0.050 | 0.025 | 0.100 | 0.075 | 0.250 |

(b) P(Satisfaction score in the 80's) = 3/40

(c) P(Satisfaction score in 80's, given participant was doctor) $= \frac{2/40}{10/40} = \frac{1}{5}$

(d) P(Participant was doctor) = 10/40

(e) P(Lawyer and score under 50) $= \frac{\text{P(Lowyer} \cap \text{Score under 50)}}{\text{P(Score under 50)}} = \frac{6}{40}$

(f) P(Score under 50 Lawyer) $= \frac{\text{P(Score under 50} \cap \text{Lawyer)}}{\text{P(Lawyer)}} = \frac{6/40}{10/40} = \frac{6}{10}$

(g) P(Satisfaction score of 70 or higher) = P(Score of 70 and above) + P(Score of 80 and above)

$$= \frac{8}{40} + \frac{3}{40} = \frac{11}{40}$$

**Example 6.22:** A market survey was conducted in four cities to find out the preference for brand A soap. The responses are shown below:

| | *Delhi* | *Kolkata* | *Chennai* | *Mumbai* |
|---|---|---|---|---|
| Yes | 45 | 55 | 60 | 50 |
| No | 35 | 45 | 35 | 45 |
| No opinion | 5 | 5 | 5 | 5 |

(a) What is the probability that a consumer selected at random, preferred brand A?
(b) What is the probability that a consumer preferred brand A and was from Chennai?

(c) What is the probability that a consumer preferred brand A given that he was from Chennai?

(d) Given that a consumer preferred brand A, what is the probability that he was from Mumbai? *[Delhi Univ., MBA, 2002; Kumaon Univ., MBA, 1999]*

**Solution:** The information from responses during market survey is as follows:

| | *Delhi* | *Kolkata* | *Chennai* | *Mumbai* | *Total* |
|---|---|---|---|---|---|
| Yes | 45 | 55 | 60 | 50 | 210 |
| No | 35 | 45 | 35 | 45 | 160 |
| No opinion | 5 | 5 | 5 | 5 | 20 |
| Total | 85 | 105 | 100 | 100 | 390 |

Let X denote the event that a consumer selected at random preferred brand A. Then

(a) The probability that a consumer selected at random preferred brand A is:

$$P(X) = 210/390 = 0.5398$$

(b) The probability that a consumer preferred brand A and was from Chennai (C) is:

$$P(X \cap C) = 60/390 = 0.1538$$

(c) The probability that a consumer preferred brand A, given that he was from Chennai:

$$P(X|C) = \frac{P(A \cap C)}{P(C)} = \frac{60/390}{100/390} = \frac{0.153}{0.256} = 0.597$$

(d) The probability that the consumer belongs to Mumbai, given that he preferred brand A

$$P(M|X) = \frac{P(M \cap X)}{P(X)} = \frac{50/390}{210/390} = \frac{0.128}{0.538} = 0.237$$

**Example 6.23:** The personnel department of a company has records which show the following analysis of its 200 engineers.

| *Age* | *Bachelor's Degree Only* | *Master's Degree* | *Total* |
|---|---|---|---|
| Under 30 | 90 | 10 | 100 |
| 30 to 40 | 20 | 30 | 50 |
| Over 40 | 40 | 10 | 50 |
| Total | 150 | 50 | 200 |

If one engineer is selected at random from the company, find:

(a) The probability that he has only a bachelor's degree.

(b) The probability that he has a master's degree, given that he is over 40.

(c) The probability that he is under 30, given that he has only a bachelor's degree. *[Kumaon Univ., MBA 1998]*

**Solution:** Let A, B, C, and D denote the events that an engineer in under 30 years of age, 40 year of age, he has bachelor's degree only, and has a master's degree, respectively. Therefore:

(a) The probability of an engineer who has only a bachelor's degree is:

$$P(C) = 150/200 = 0.75$$

(b) The probability of an engineer who has a master's degree, given that he is over 40 years is:

$$P(D|B) = \frac{P(D \cap B)}{P(B)} = \frac{10/200}{50/200} = \frac{10}{50} = 0.20$$

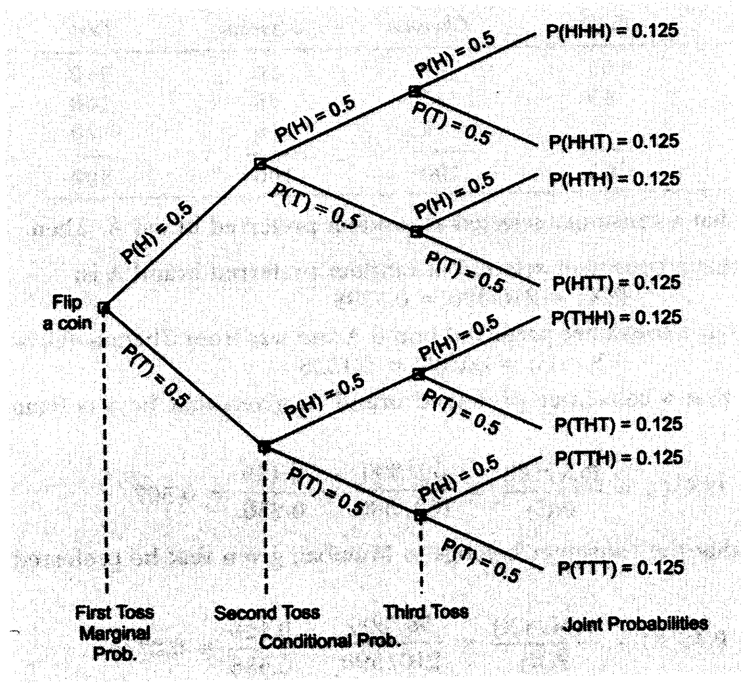
(c) The probability of an engineer who is under 30 years, given that he has only bachelor's degree is:

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{90/200}{150/200} = \frac{90}{150} = 0.60$$

## 6.6 PROBABILITY TREE DIAGRAM

Decision-makers at times face difficulty in constructing the joint probability table. Thus they prefer to use the *probability tree* to probability calculations. The probability tree diagram for tossing a coin three consecutive times is shown in Fig. 6.5.

Figure 6.5
Probability Tree Diagram




The probability tree diagram is convenient for calculating joint probabilities when events occur at different times or stages. In probability trees, time moves from left to right. The complete tree exhibits each outcome as a *single path* from beginning to end. Each path corresponds to a distinct joint event. A joint event is represented by a path through the tree and its probability is determined by multiplying all the individual branch probabilities for its path.

In this example of three tosses of a coin, at each toss the probability of either event's (H or T) occurring remains the same, that is, the events are independent. The joint probabilities of events occurring in succession are calculated by multiplying the probabilities of each event.

The events emanating from a single breaking point are mutually exclusive and collectively exhaustive, so that exactly one must occur. All the probabilities on branches within the same fork must therefore sum to 1.

In this case the results in the diagram should not be confused with conditional probabilities. The probability of a head and then two tails occurring on three consecutive tosses is computed prior to any tosses taking place. If the first two tosses have already occurred, then the probability of getting a tail on the third toss is still 0.5, that is, $P(T | HT) = P(T) = 0.5$.

**Example 6.24:** Each salesperson is rated either below average, average, or above average with respect to sales ability. Each of them is also rated with respect to his or her potential for advancement-either fair, good or excellent. These traits of the 500 sales person are given below:

| | *Potential for Advancement* | | |
|---|---|---|---|
| *Sales Ability* | *Fair* | *Good* | *Excellent* |
| Below average | 16 | 12 | 22 |
| Average | 45 | 60 | 45 |
| Above average | 93 | 72 | 135 |

(a) What is the probability that a randomly selected salesperson will have above average sales ability and excellent potential for advancement?

(b) Construct a tree diagram showing all the probabilities. Conditional probabilities and joint probabilities.

**Solution:** (a) Let A and B represent the event that a salesperson will have above average sales ability and excellent potential for advancement. The joint probability of these traits is given by

$$P(A \text{ and } B) = P(A) \times P(B) = \frac{300}{500} \times \frac{135}{300} = 0.27 +$$

(b) The tree diagram of probabilities is shown in Fig. 6.6.

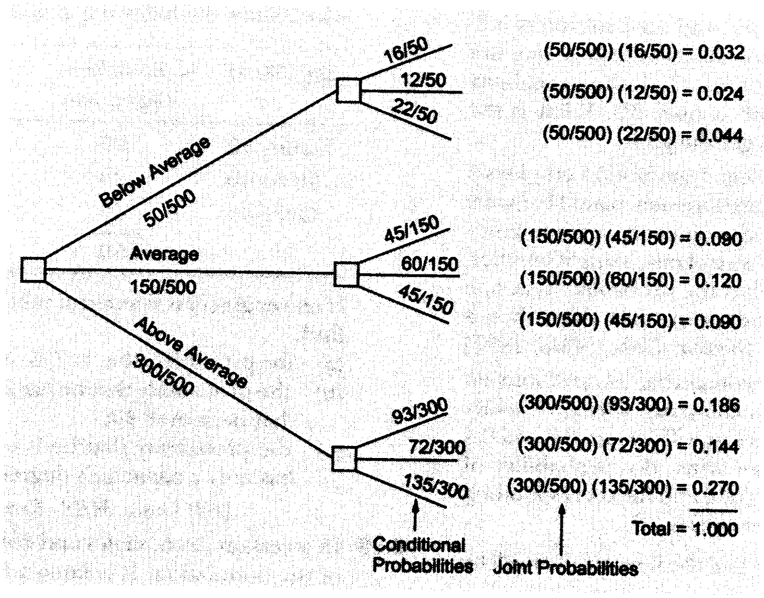



**Figure 6.6**
Tree Diagram

## Self-Practice Problems 6B

**6.10** Mr. X has 2 shares in a lottery in which there are 2 prizes and 5 blanks. Mr. Y has 1 share in a lottery in which there is 1 prize and 2 blanks. Show that the chance of Mr. X's success to that of Mr. Y's as 15 : 7.

**6.11** Explain whether or not each of the following claims could be correct:

(a) A businessman claims the probability that he will get contract A is 0.15 and that he will get contract B is 0.20. Furthermore, he claims that the probability of getting A or B is 0.50.

(b) A market analyst claims that the probability of selling ten million rupees of plastic A or five million rupees of plastic B is 0.60. He also claims that the probability of selling ten million rupees of A and five million rupees of B is 0.45.

**6.12** The probability is 0.3 that an applicant for a Management Accountant's job has a postgraduate degree, 0.7 that he has had some work experience as a chief Financial Accountant, and 0.2 that he has both. Out of 300 applicants, approximately, what number would have either a postgraduate degree or some professional work experience?

**6.13** A can hit a target 3 times in 5 shots; B, 2 times in 5 shots; C, 3 times in 4 shots. They fire a volley. What is the probability that 2 shots hit?

**6.14** A problem in business statistics is given to five students, A, B, C, D, and E. Their chances of solving it are 1/2, 1/3, 1/4, 1/5, and 1/6 respectively. What is the probability that the problem will be solved?

[*Madras Univ., BCom, 1996; Kumaon Univ., MBA, 2000*]

**6.15** A husband and wife appear in an interview for two vacancies for the same post. The probability of husband's selection is 1/7 and that of wife's selection is 1/5. What is the probability that

(a) only one of them will be selected?

(b) both of them will be selected?

(c) none of them will be selected?

**6.16** A candidate is selected for interviews for three posts. For the first there are 3 candidates, for the second there are 4, and for the third there are 2. What is the probability of his getting selected for at least one post?

**6.17** Three persons A, B, and C are being considered for appointment as Vice-Chancellor of a university, and whose chances of being selected for the post are in the proportion 14 : 2 : 3 respectively. The probability that if A is selected, will introduce democratization in the university structure is 0.3, and the corresponding probabilities for B and C doing the same are respectively 0.5 and 0.8. What is the probability that democratization would be introduced in the university?

**6.18** There are three brands, say X, Y, and Z of an item available in the market. A consumer chooses exactly one of them for his use. He never buys two or more brands simultaneously. The probabilities that he buys brands X, Y, and Z are 0.20, 0.16, and 0.45.

(a) What is the probability that he does not buy any of the brands?

(b) Given that a customer buys some brand, what is the probability that he buys brand X?

**6.19** There is 50-50 chance that a contractor's firm, A, will bid for the construction of a multi-storeyed building. Another firm, B, submits a bid and the probability is 3/5 that it will get the job, provided that firm A does not submit a bid. If firm A submits a bid, the probability that firm B will get the job is only 2/3. What is the probability that firm B will get the job?

**6.20** Plant I of XYZ manufacturing organization employs 5 production and 3 maintenance foremen, plant II of same organization employs 4 production and 5 maintenance foremen. From any one of these plants, a single selection of two foremen is made. Find the probability that one of them would be a production and the other a maintenance foreman. *[Bombay Univ., MMS, 1997]*

**6.21** Two sets of candidates are competing for positions on the board of directors of a company. The probability that the the first and second sets will win are 0.6 and 0.4 respectively. If the first set wins, the probability of introducing a new product is 0.8 and the corresponding probability if the second set wins is 0.3.

(a) What is the probability that the new product will be introduced?

(b) If the new product was introduced, what is the probability that the first set won as directors?

**6.22** If a machine is correctly set up, it will produce 90 per cent acceptable items. If it is incorrectly setup, it will produce 40 per cent acceptable items. Past experience shows that 80 per cent of the setups are correctly done. If after a certain setup, the machine produces 2 acceptable items as the first 2 pieces, find the probability that the machine is correctly set up. *[Delhi Univ., BCom, (Hons), 1998]*

**6.23** A firm plans to bid Rs 300 per tonne for a contract to supply 1,000 tonnes of a metal. It has two competitors A and B. It assumes the probability of A bidding less than Rs 300 per tonne to be 0.3 and B's bid to be less than Rs 300 per tonne to be 0.7. If the lowest bidder gets all the business and the firms bid independently, what is the expected value of the contract to the firm?

**6.24** An investment consultant predicts that the odds against the price of a certain stock going up during the next week are 2 : 1 and odds in favour of the price remaining the same are 1 : 3. What is the probability that the price of the stock will go down during the next week?

**6.25** An article manufactured by a company consists of two parts A and B. In the process of manufacture of part A, 9 out of 100 are likely to be defective. Similarly, 5 out of 100 are likely to be defective in the manufacture of part B. Calculate the probability that the assembled part will not be defective.

**6.26** A product is assembled from three components X, Y, and Z, the probability of these components being defective is respectively 0.01, 0.02, and 0.05. What is the probability that the assembled product will not be defective?

**6.27** The daily production of a machine producing a very complicated item gives the following probabilities for the number of items produced: P(1) = 0.20, P(2) = 0.35, and P(3) = 0.45. Furthermore, the probability of defective items being produced is 0.02. Defective items are assumed to occur independently. Determine the probability of no defectives during a day's production.

**6.28** The personnel department of a company has records which show the following analysis of its 200 engineers:

| *Age (Years)* | *Bachelor's Degree only* | *Master's Degree* | *Total* |
|---|---|---|---|
| Under 30 | 90 | 10 | 100 |
| 30 to 40 | 20 | 30 | 50 |
| Over 40 | 40 | 10 | 50 |
| | 150 | 50 | 200 |

If one engineer is selected at random from the company, find:

(a) the probability that he has only a bachelor's degree;

(b) the probability that he has a master's degree given that he is over 40;

(c) the probability that he is under 30 given that he has only a bachelor's degree.

*[HP Univ., MBA; Kumaon Univ., MBA 1998]*

**6.29** In a certain town, males and females form 50 per cent of the population. It is known that 20 per cent of the males and 5 per cent of the females are unemployed. A research student studying the employment situation selects unemployed persons at random. What is the probability that the person selected is (a) male, (b) female?

*[Delhi Univ. MCom, 1999; Kumaon Univ., MBA, 1998]*

**6.30** You note that your officer is happy in 60 per cent cases of your calls. You have also noticed that if he is happy, he accedes to your requests with a probability of 0.4, whereas if he is not happy, he accedes to your requests with a probability of 0.1. You call on him one day and he accedes to your request. What is the probability of his being happy? *[HP, MBA, 1996]*

**6.31** In a telephone survey of 1000 adults, respondents were asked about the expenses on a management education and the relative necessity of some form of financial assitance. The respondents were classified according to whether they currently had a child studying in a school of management and whether they thought that the loan burden for most management students is: too high, right amount, or too little. The proportions responding in each category are given below.

| | *Too High (A)* | *Right Amount (B)* | *Too Little (C)* |
|---|---|---|---|
| Child studying management (D) : | 0.35 | 0.08 | 0.01 |
| No child studying management (E) : | 0.25 | 0.20 | 0.11 |

Suppose one respondent is chosen at random from this group. Then

(a) What is the probability that the respondent has a child studying management.

(b) Given that the respondent has a child studying management, what is the probability that he/she ranks the loan burden as 'too high'.

(c) Are events D and A independent? Explain.

**6.32** In a colour preference experiment, eight toys are placed in a container. The toys are identical except for colour-two are red, and six are green. A child is asked to choose two toys at random. What is the probability that the child chooses the two red toys?

**6.33** A survey of executives dealt with their loyalty to the company. One of the questions was, 'If you were given an offer by another company equal to or slightly better than your present position, would you remain with the company?' The responses of 200 executives in the survey cross-classified with their length of service with the company are shown below:

| | *Length of Service* | | | | |
|---|---|---|---|---|---|
| *Loyalty* | *Less than 1 year* | *1–5 years* | *6–10 years* | *More than 10 years* | *Total* |
| Would remain : | 10 | 30 | 5 | 75 | 120 |
| Would not remain : | 25 | 15 | 10 | 30 | 80 |

What is the probability of randomly selecting an executive who is loyal to the company (would remain) and who has more than 10 years of service.

## Hints and Answers

**6.10** Considering Mr. X's chances of success.

A = event that 1 share brings a prize and 1 share goes blank.

B = event that both the shares bring prizes.

C = event that X succeeds in getting atleast one prize $= A \cup B$.

Since A and B are mutually exclusive, therefore

$$P(C) = P(A \cup B) = P(A) + P(B) = \frac{{}^2C_1 \times {}^5C_1}{{}^7C_2} + \frac{{}^2C_2 \times {}^5C_0}{{}^7C_2}$$

Similarly, if D denotes the event that Y succeeds in getting a prize, then we have

$$P(D) = \frac{{}^1C_1}{{}^3C_1} = \frac{1}{3}$$

$$\text{X's chance of success: Y's chance of success} = \frac{15}{21} : \frac{1}{3} = 15 : 7.$$

**6.11** (a) $P(A \cap B) = -0.15$

(b) $P(A) + P(B) = 1.05$

**6.12** Let A = Applicant has P.G degree; B = Applicant has work experience;

Given, $P(A) = 0.3$, $P(B) = 0.7$, and $P(A \cap B) = 0.2$. Therefore

$$300 \times P(A \cup B) = 300[P(A) + P(B) - P(A \cap B)] = 240$$

**6.13** The required event that two shots may hit the target, can happen in the following mutually exclusive cases:

(i) A and B hit and C fails to hit the target

(ii) A and C hit and B fails to hit the target

(iii) B and C hit and A fails to hit the target

Hence, the required probability that any two shots hit is given by, $P = P(i) + P(ii) + P(iii)$.

Let $E_1$, $E_2$, and $E_3$ be the event of hitting the target by A, B, and C respectively. Therefore

$$P(i) = P(E_1 \cap E_2 \cap \bar{E}_3) = P(E_1) \cdot P(E_2) \cdot P(\bar{E}_3) = \left(\frac{3}{5}\right)\cdot\left(\frac{2}{5}\right)\cdot\left(1-\frac{3}{4}\right) = \frac{6}{100}$$

$$P(ii) = P(E_1 \cap \bar{E}_2 \cap E_3) = \left(\frac{3}{5}\right)\left(1-\frac{2}{5}\right)\left(\frac{3}{4}\right) = \frac{27}{100}$$

$$P(iii) = P(\bar{E}_1 \cap E_2 \cap E_3) = \left(1-\frac{3}{5}\right)\left(\frac{2}{5}\right)\left(\frac{3}{4}\right) = \frac{12}{100}$$

Since all the three events are mutually exclusive events, hence the required probability is given by

$$P(i) + P(ii) + P(iii) = \frac{6}{100} + \frac{27}{100} + \frac{12}{100} = \frac{9}{20}$$

**6.14** P(problem will be solved)

= 1 – P(problem is not solved)

= 1 – P(all students fail to solve the problem)

$$= 1 - P(\bar{A} \cap \bar{B} \cap \bar{C} \cap \bar{D} \cap \bar{E})$$

$$= 1 - P(\bar{A})\, P(\bar{B})\, P(\bar{C})\, P(\bar{D})\, P(\bar{E})$$

$$= 1 - \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{6}\right)$$

$$= 1 - \frac{1}{6} = \frac{5}{6}$$

**6.15** P(only one of them will be selected)

$$= P(H \cap \bar{W}) \cup (\bar{H} \cap W) = P(H \cap \bar{W}) + (\bar{H} \cap W)$$

$$= P(H)\, P(\bar{W}) + P(\bar{H})\, P(W)$$

$$= \frac{1}{7}\left(1-\frac{1}{5}\right) + \left(1-\frac{1}{7}\right)\frac{1}{5} = \frac{2}{7}$$

(b) P(both of them will be selected)

$$P(H \cap W) = P(H) \cdot P(W) = \frac{1}{35}$$

(c) P(none of them will be selected)

$$P(\bar{H} \cap \bar{W}) = P(\bar{H}) \cdot P(\bar{W}) = \frac{24}{35}$$

**6.17** P(D) = Prob. of the event that democratization would be introduced

$$= P[(A \cap D) \cup [(B \cap D) \cup (C \cap D)]$$

$$= P[(A \cap D) + P(B \cap D) + P(C \cap D)$$

$= P(A) \cdot P(D \mid A) + P(B) \cdot P(D \mid B) + P(C) \cdot P(D \mid C)$

$$= 0.3\left(\frac{4}{9}\right)+0.5\left(\frac{2}{9}\right)+0.8\left(\frac{3}{9}\right) = 0.51$$

**6.18** (a) P(customer does not buy any brand)

$= P[\bar{X} \cap \bar{Y} \cap \bar{Z}] = 1 - P[(X \cup Y \cup Z]$
$= 1 - [P(X) + P(Y) + P(Z)]$
$= 1 - [0.20 + 0.16 + 0.45] = 0.19$

(b) P(customer buys brand X) $= P[X \mid (X \cup Y \cup Z)]$

$$= \frac{P[X \cap (X \cup Y \cup Z)]}{P(X \cup Y \cup Z)}$$

$$= \frac{P(X)}{P(X)+P(Y)+P(Z)-P(X\cap Y)-P(Y\cap Z)-P(X\cap Z)+P(X\cap Y\cap Z)}$$

$$= \frac{0.2}{0.2+0.16+0.45-0-0-0-0} = 0.247$$

**6.19** $P(A) = 1/2$, $P(B \mid A) = 2/3$, and $P(B \mid \bar{A}) = 3/5$.

$P(B) = P(A \cap B) + P(\bar{A} \cap B)$

$= P(A) \cdot P(B \mid A) + P(\bar{A}) \cdot P(B \mid \bar{A})$

$$= \frac{1}{2}\cdot\frac{2}{3}+\frac{1}{2}\cdot\frac{3}{5} = \frac{19}{30}.$$

**6.20** Let $E_1$ and $E_2$ be the events that Plant I and II is selected respectively. Then the probability of the event E that in a batch of 2, one is the production and the other is the maintenance man is

$P(E) = P(E_1 \cap E) + P(E_2 \cap E)$
$= P(E_1) \cdot P(E \mid E_1) + P(E_2) \cdot P(E \mid E_2)$

$$= \frac{1}{2}\cdot\frac{{}^5C_1 \cdot {}^3C_1}{{}^8C_2}+\frac{1}{2}\cdot\frac{{}^4C_1 \cdot {}^5C_1}{{}^9C_2}$$

$$= \frac{1}{2}\cdot\frac{15}{28}+\frac{1}{2}\cdot\frac{5}{9} = \frac{275}{504}$$

**6.22** Let A = event that the item is acceptable;

$B_1$ and $B_2$ = events that machine is correctly and incorrectly setup, respectively.

Given, $P(A \mid B_1) = 0.9$; $P(A \mid B_2) = 0.4$ ; $P(B_1) = 0.8$ and $P(B_2) = 0.2$. Then $P(B_1 \mid A) = 0.9$.

**6.23** There are two competitors A and B and the lowest bidder gets the contract.

Value of plan $= 300 \times 1,000 = 3,00,000$

Contractor A: P(Bid < 300) = 0.3;
P(Bid ≥ 300) = 0.7

Contractor B: P(Bid < 300) = 0.7;
P(Bid ≥ 300) = 0.3

(i) If both bids are less than Rs 300, probability is $0.3 \times 0.7 = 0.21$. Therefore plan value is:
$3,00,000 \times 0.21 = 63,000$.

(ii) If A bids less than 300 and B bids more than 300, probability is $0.3 \times 0.3 = 0.9$. Therefore, plan value is: $3,00,000 \times 0.09 = 27,000$.

(iii) B bids less than 300 while A bids more than 300, probability is: $0.7 \times 0.7 = 0.49$. Therefore plan value is: $3,00,000 \times 0.49 = 1,47,000$.

Therefore, expected value of plan is
$63,000 + 27,000 + 1,47,000 = 2,37,000$.

**6.24** P(price of a certain stock not going up) = 2/3
P(price of a certain stock remaining same) = 1/4
The probability that the price of the stock will go down during the next week

= P(price of the stock not going up and not remaining same)

= P(price of the stock not going up × P(price of the stock not remaining same)

$$= \left(\frac{2}{3}\right)\times\left(1-\frac{1}{4}\right) = \left(\frac{2}{3}\right)\times\left(\frac{3}{4}\right) = \frac{1}{2} = 0.5$$

**6.25** The assembled part will be defective if any of the parts is defective.

The probability of the assembled part being defective:
= P[Any of the part is defective]
$= P[A \cup B] = P(A) + P(B) - P(AB)$

$$= \frac{9}{100}+\frac{5}{100}-\left(\frac{9}{100}\right)\times\left(\frac{5}{100}\right) = 0.1355$$

The probability that assembled part is not defective
$= 1 - 0.1355 = 0.8645$.

**6.26** Let A, B, and C denote the respective probabilities of components X, Y, and Z being defective.

$P(A) = 0.01,\ P(B) = 0.02,\ P(C) = 0.05$

P(A or B or C) $= P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(ABC)$
$= 0.01 + 0.02 + 0.05 - 0.0002 - 0.0010 - 0.0005 + 0.00001 = 0.0784$

Hence the probability that the assembled product will not be defective = 1 – 0.0784 or 0.9216.

**6.27** Let A be the event that no defective item is produced during a day. Then

$P(A) = P(I) \cdot P(A \mid I) + P(2) \cdot P(A \mid 2) + P(3) \cdot P(A \mid 3)$

The probability that a defective item is produced = 0.02. Probability that a non-defective item is produced = 1 – 0.02 = 0.98. Also defectives are assumed to occur independently, therefore:

$P(A \mid I) = 0.98$, $P(A \mid 2) = (0.98)(0.98)$ and
$P(A \mid 3) = (0.98)(0.98)(0.98)$

$P(A) = (0.20(0.98) + (0.35)(0.98)^2 + (0.45)(0.98)^3$
$= 0.1960 + 0.3361 + 0.4322 = 0.9643$

Hence the probability of no defectives during a day's production is 0.9643.

**6.28** A: an engineer has a bachelor's degree only
B: an engineer has a master's degree
C: an engineer is under 30 years of age
D: an engineer is over 40 years of age

(a) $P(A) = 150/200 = 0.75$

(b) $P(B \mid D) = \frac{P(B \cap D)}{P(D)} = \frac{10/200}{50/200} = 0.20$

(c) $P(C \mid A) = \frac{P(C \cap A)}{P(A)} = \frac{90/200}{150/200} = 0.60$

**6.29** Given that

| | *Employed* | *Unemployed* | *Total* |
|---|---|---|---|
| Males | 0.40 | 0.10 | 0.50 |
| Females | 0.475 | 0.025 | 0.50 |
| Total | 0.875 | 0.125 | 1.00 |

Let M and F be the male and female chosen, respectively.
U = Male, female chosen is unemployed

(a) $P(M \mid U) = \dfrac{P(M \cap U)}{P(U)} = \dfrac{0.10}{0.125} = 0.80$

(b) $P(F \mid U) = \dfrac{P(F \cap U)}{P(U)} = 0.20$

**6.30** The probability that the officer is happy and accedes to requests = 0.6 × 0.4.

The probability that the officer is unhappy and accedes to requests = 0.4 × 0.1 = 0.04.

Total probability of acceding to requests = 0.24 + 0.04 = 0.28.

The probability of his being happy if he accedes to requests = 0.24/0.28 = 0.875.

**6.31** (a) P(D) = P(A) + P(B) + P(C) = 0.35 + 0.08 + 0.01 = 0.44

(b) $P(A \mid D) = \dfrac{P(A \cap D)}{P(D)} = \dfrac{0.35}{0.44} = 0.80$

(c) Since P(A | D) = 0.80 and P(A) = 0.35 + 0.25 = 0.80, events A and D must be independent.

**6.32** Let R = Red toy is chosen and G = Green toy is chosen.

P(Both toys are R) = P(R on first choice ∩ R on second choice)
= P(R on first choice) P(R on second choce | R on first choice)
= (2/8( (1/7) = 1/28.

**6.33** Let A : Executive who would remain with the company despite an equal or slightly better offer

B : Executive who has more than 10 years of service with the company

P(A and B)=P(A) P(B|A)=(120/200) (75/120) = 0.375

## 6.7 BAYES' THEOREM

In the 18th Century Reverend Thomas Bayes, an English Presbyterian minister, raised a question: Does God really exist? To answer this question, he attempted to develop a formula to determine the probability that God does exist based on evidence that was available to him on earth. Later Laplace refined Bayes' work and gave it the name *Bayes' Theorem*.

The **Bayes' theorem** is useful in revising the original probability estimates of known outcomes as we gain additional information about these outcomes. The prior probabilities, when changed in the light of new information, are called *revised* or *posterior probabilities*.

**Bayes' theorem:** A method to compute posterior probabilities (conditional probabilities under statistical dependence).

Suppose $A_1, A_2, \ldots, A_n$ represent $n$ mutually exclusive and collectively exhaustive events with prior marginal probabilities $P(A_1), P(A_2), \ldots, P(A_n)$. Let B be an arbitrary event with $P(B) \neq 0$ for which conditional probabilities $P(B \mid A_1), P(B \mid A_2), \ldots, P(B \mid A_n)$ are also known. Given the information that outcome B has occured, the revised (or posterior) probabilities $P(A_i \mid B)$ are determined with the help of Bayes' theorem using the formula:

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)} \qquad (6\text{-}8)$$

where the posterior probability of events Ai given event B is the conditional probability $P(A_i|B)$

Since events $A_1, A_2, \ldots, A_n$ are mutually exclusive and collectively exhaustive, the event B is bound to occur with either $A_1, A_2, \ldots, A_n$. That is,

$$B = (A_1 \cap B) \cup (A_2 \cap B) \cup \ldots \cup (A_n \cap B)$$

where the **posterior probability** of $A_i$ given B is the conditonal probability $P(A_i|B)$.

**Posterior probability:** A revised probability of an event obtained after getting additional information.

Since $(A_1 \cap B), (A_2 \cap B) \ldots (A_n \cap B)$ are mutually exclusive, we get

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \ldots + P(A_n \cap B) = \sum_{i=1}^{n} P(A_i \cap B)$$

$$= P(A_1)\,P(B|A_1) + P(A_2)\,P(B|A_2) + \ldots + P(A_n)\,P(B|A_n)$$

$$= \sum_{i=1}^{n} P(A_i)\,P(B|A_i)$$

From formula (6-8) for a fixed $i$, we have

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i) . P(A_i)}{P(B)}$$

$$= \frac{P(B|A_i) . P(A_i)}{P(A_1) . P(B|A_1) + P(A_2) P(B|A_2) + \ldots + P(A_n) P(B|A_n)}$$

**Example 6.25:** Suppose an item is manufacture by three machines X, Y, and Z. All the three machines have equal capacity and are operated at the same rate. It is known that the percentages of defective items produced by X, Y, and Z are 2, 7, and 12 per cent respectively. All the items produced by X, Y, and Z are put into one bin. From this bin, one item is drawn at random and is found to be defective. What is the probability that this item was produced on Y?

**Solution:** Let A be the defective item. We know the prior probability of defective items produced on X, Y, and Z, that is, P(X) = 1/3; P(Y) = 1/3 and P(Z) = 1/3. We also know that

$$P(A|X) = 0.02, \quad P(A|Y) = 0.07, \quad P(A|Z) = 0.12$$

Now having known that the item drawn is defective, we want to know the probability that it was produced by Y. That is

$$P(Y|A) = \frac{P(A|Y) . P(Y)}{P(X) . P(A|X) + P(Y) . P(A|Y) + P(Z) . P(A|Z)}$$

$$= \frac{(0.07) . (1/3)}{(1/3)(0.02) + (1/3)(0.07) + (1/3)(0.12)} = 0.33$$

**Example 6.26:** Assume that a factory has two machines. Past records show that machine 1 produces 30 per cent of the items of output and machine 2 produces 70 per cent of the items. Further 5 per cent of the items produced by machine 1 were defective and only 1 per cent produced by machine 2 were defective. If a defective item is drawn at random, what is the probability that the defective item was produced by machine 1 or machine 2?

**Solution:** Let $A_1$ = Event of drawing an item produced by machine 1,
$A_2$ = Event of drawing an item produced by machine 2,
and $D$ = Event of drawing a defective item produced either by machine 1 or machine 2.

From the data in the problem, we know that

$$P(A_1) = 0.30, \quad P(A_2) = 0.70; \quad P(D \mid A_1) = 0.05, \quad P(D \mid A_2) = 0.1$$

The data of the problem can now be summarized as under:

| *Event* (1) | *Prior Probability* $P(A_i)$ (2) | *Conditional Probability Event* $P(D \mid A_i)$ (3) | *Joint Probability* $P(A_i \text{ and } D)$ (2) × (3) | *Posterior (revised) Probability* $P(A_i \mid D) P(A_i \text{ and } D)$ |
|---|---|---|---|---|
| $A_1$ | 0.30 | 0.05 | 0.015 | 0.015/0.022 = 0.682 |
| $A_2$ | 0.70 | 0.01 | 0.007 | 0.007/0.022 = 0.318 |

Here $$P(D) = \sum_{i=1}^{2} P(D|A_i) P(A_i) = 0.05 \times 0.30 + 0.01 \times 0.70 = 0.22$$

From the above table, the probability that the defective item was produced by machine 1 is 0.682 or 68.2 per cent and that by machine 2 is only 0.318 or 31.8 per cent. We may now say that the defective item is more likely drawn from the output produced by machine 1.

**Example 6.27:** A company uses a 'selling aptitude test' in the selection of salesmen. Past experience has shown that only 70 per cent of all persons applying for a sales position achieved a classification 'dissatisfactory' in actual selling, whereas the remainder were classified as 'satisfactory', 85 per cent had scored a passing grade in the aptitude test. Only 25 per cent of those classified dissatisfactory, had passed the test on the basis of this information. What is the probability that a candidate would be a satisfactory salesman given that he passed the aptitude test?

**Solution:** Let A and B be the event representing 'unsatisfactory' classification as a salesman and 'passing the test', respectively. Now the probability that a candidate would be 'satisfactory' salesman given that he passed the aptitude test is:

$$P(A|B) = \frac{(0.70)(0.85)}{(0.70)(0.85) + (0.30)(0.25)} = \frac{0.595}{0.595 + 0.075} = 0.888$$

Assuming no change in the type of candidates applying for the selling positions, the probability that a random applicant would be satisfactory is 70 per cent. On the other hand, if the company only accepts an applicant if he passed the test, the probability increases to 88.8 per cent.

**Example 6.28:** In a bolt factory machines A, B, and C manufacture respectively 25 per cent, 35 per cent and 40 per cent of the total output. Of the total of their output 5, 4, and 2 per cent are defective bolts, A bolt is drawn at random and is found to be defective. What is the probability that it was manufactured by machines A, B, or C?

*[Punjab Univ., MCom, Madurai Univ., MCom, 1998]*

**Solution:** Let, $A_i$ $(i = 1, 2, 3)$ be the event of drawing a bolt produced by machine A, B, and C, respectively. From the data we know that

$$P(A_1) = 0.25;\ P(A_2) = 0.35, \text{ and } P(A_3) = 0.40$$

From the additional information, we know that

B = the event of drawing a defective bolt

Thus, $P(B|A_1) = 0.05$; $P(B|A_2) = 0.04$; and $P(B|A_3) = 0.02$

The table of posterior probabilities can be prepared as under:

| *Event* | *Prior Probability* $P(A_i)$ | *Conditional Probability* $P(B \mid A_i)$ | *Joint Probability* (2) × (3) | *Posterior Probability* |
|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) |
| $A_1$ | 0.25 | 0.05 | 0.0125 | 0.0125 ÷ 0.0345 = 0.362 |
| $A_2$ | 0.35 | 0.04 | 0.0140 | 0.014 ÷ 0.0345 = 0.406 |
| $A_3$ | 0.40 | 0.02 | 0.0080 | 0.008 ÷ 0.0345 = 0.232 |
| Total | 1.00 | | 0.0345 | 1.000 |

The above table shows the probability that the item was defective and produced by machine A is 0.362, by machine B is 0.406, and machine C is 0.232.

## Self-Practice Problems 6C

**6.34** A manufacturing firm produces steel pipes in three plants with daily production volumes of 500, 1000, and 2,000 units respectively. According to past experience, it is known that the fractions of defective output produced by the three plants are respectively 0.005, 0.008, and 0.010. If a pipe is selected from a day's total production and found to be defective, find out (a) from which plant the pipe comes, (b) what is the probability that it came from the first plant? *[IIT Roorkee MBA, 2004]*

**6.35** In a post office, three clerks are assigned to process incoming mail. The first clerk, A, processes 40 per cent; the second clerk, B, processes 35 per cent; and the third clerk, C, processes 25 per cent of the mail. The first clerk has an error rate of 0.04, the second has an error rate of 0.06, and the third has an error rate of 0.03. A mail selected at random from a day's output is found to have an error. The postmaster wishes to know the probability that it was processed by clerk A or clerk B or clerk C.

**6.36** A certain production process produces items 10 per cent of which defective. Each item is inspected before supplying to customers but 10 per cent of the time the inspector incorrectly classifies an item. Only items classified as good are supplied. If 820 items have been supplied in all, how many of them are expected to be defective?

**6.37** A factory produces certain types of output by three machines. The respective daily production figures are: Machine A = 3000 units; Machine B = 2500 units; and Machine C = 4500 units. Past experience shows that 1 per cent of the output produced by machine A is defective. The corresponding fractions of defectives for the other two machines are 1.2 and 2 per cent respectively. An item is drawn at random from the day's production and is found to be defective. What is probability that it comes from the output of (a) Machine A; (b) Machine B; (c) Machine C?

**6.38** In a bolt factory machines A, B, and C manufacture respectively 25 per cent, 30 per cent and 40 per cent of the total output. Of the total of their output 5, 4, and 2 per cent are defective bolts. A bolt is drawn at random from the lot and is found to be defective. What are the probabilities that it was manufactured by machines A, B, or C?

**6.39** In a factory manufacturing pens, machines X, Y, and Z manufacture 30, 30, and 40 per cent of the total production of pens, respectively. Of their output 4, 5, and 10 per cent of the pens are defective. If one pen is selected at random, and it is found to be defective, what is the probability that it is manufactured by machine Z?

**6.40** A worker-operated machine produces a defective item with probability 0.01, if the worker follows the machine's operating instruction exactly, and with probability 0.03 if he does not. If the worker follows the instructions 90 per cent of the time, what proportion of all items produced by the machine will be defective?

**6.41** Medical case histories indicate that different illnesses may produce identical symptoms. Suppose a particular set of symptoms, 'H' occurs only when one of three illnesses: A, B or C occurs, with probabilities 0.01, 0.005 and 0.02 respectively. The probability of developing the symptoms H, given a illness A, B and C are 0.90, 0.95 and 0.75 respectively. Assuming that an ill person shows the symptoms H, what is the probability that a person has illness A?

## Hints and Answers

**6.34** Let $A_1$, $A_2$ and $A_3$ = production volume of plant I, II, and III, respectively.

$$E = \text{defective steel pipe}$$

$$P(A_1) = 500/3500 = 0.1428;$$
$$P(A_2) = 1000/3500 = 0.2857;$$
$$P(A_3) = 2000/3500 = 0.5714$$
$$P(E \mid A_1) = 0.005,\ P(E \mid A_2) = 0.008,$$
$$\text{and } P(E \mid A_3) = 0.010.$$
$$P(A_1 \cap E) = P(A_1)\,P(E \mid A_1) = 0.1428 \times 0.005 = 0.0007;$$
$$P(A_2 \cap E) = P(A_2)\,P(E \mid A_2) = 0.2857 \times 0.008 = 0.0022$$
$$P(A_3 \cap E) = P(A_3)\,P(E \mid A_3) = 0.5714 \times 0.010 = 0.057$$
$$P(E) = P(A_1 \cap E) + P(A_2 \cap E) + P(A_3 \cap E) = 0.0007 + 0.0022 + 0.057 = 0.0599$$

(a) $$P(A_1 \mid E) = \frac{P(A_1 \cap E)}{P(E)} = \frac{0.0007}{0.0599} = 0.0116$$

$$P(A_2 \mid E) = \frac{P(A_2 \cap E)}{P(E)} = \frac{0.0022}{0.0599} = 0.0367;$$

$$P(A_3 \mid E) = \frac{P(A_3 \cap E)}{P(E)} = \frac{0.057}{0.0599} = 0.951$$

Since $P(A_3 \mid E)$ is highest, the defective steel pipe has most likely come from the third plant

(b) $$P(A_1 \mid E) = \frac{P(A_1 \cap E)}{P(E)} = \frac{P(A_1)\,P(E \mid A_1)}{P(E)}$$
$$= \frac{(500/3500) \times 0.005}{0.0599} = 0.0119$$

**6.35** Let A, B, and C = mail processed by first, second, and third clerk, respectively

E = mail containing error

Given $P(A) = 0.40$, $P(B) = 0.35$, and $P(C) = 0.25$
$P(E \mid A) = 0.04$, $P(E \mid B) = 0.06$,
and $P(E \mid C) = 0.03$

$$\therefore P(A \mid E) = \frac{P(A)\,P(E \mid A)}{P(E)}$$
$$= \frac{P(A)\,P(E \mid A)}{P(A)\,P(E \mid A) + P(B)\,P(E \mid B) + P(C)\,P(E \mid C)}$$
$$= \frac{0.40 \times 0.04}{0.40\,(0.04) + 0.35\,(0.06) + 0.25\,(0.03)} = 0.36$$

Similarly $P(B \mid E) = [P(B)\,P(E \mid B)]/P(E) = 0.47$
$P(C \mid E) = [P(C)\,P(E \mid C)]/P(E) = 0.17$

**6.36** $P(D)$ = Probability of defective item = 0.1; P(classified as good | defective) = 0.1

$\therefore$ $P(G)$ = Probability of good item = $1 - P(D) = 1 - 0.1 = 0.9$

P(classified as good | good) = 1 – P(classified as good | defective) = 1 – 0.1 = 0.9

$\therefore$ P(defective | classified as good)

$$= \frac{P(D) \cdot P(\text{classified as good} \mid \text{defective})}{[P(D) \cdot P(\text{classified as good} \mid D) + P(G)\,P(\text{classified as good} \mid G)]}$$

$$= \frac{0.1 \times 0.1}{0.1 \times 0.1 + 0.9 \times 0.9} = \frac{0.01}{0.82} = 0.012.$$

**6.37** (a) 0.20 (b) 0.20 (c) 0.60

**6.38** $P(A) = 0.37$, $P(B) = 0.40$, $P(C) = 0.23$

**6.39** $P(Z) = 0.6639$

**6.40** $P(A) = 0.012$

**6.41** $P(A \mid H) = 0.3130$

# Formulae Used

1. Counting methods for determining the number of outcomes
   - Multiplication method
     (i) $n_1 \times n_2 \times \ldots \times n_k$
     (ii) $n_1 \times n_2 \times \ldots \times n_k = n^k$
     when the event in each trial is the same
   - Number of Permutations $^nP_r = \frac{n!}{(n-r)!}$
   - Number of Combinations $^nC_r = \frac{n!}{r!\,(n-r)!}$
2. Classical or *a priori* approach of computing probability of an event A
   $$P(A) = \frac{\text{Number of favourable cases for A}}{\text{All possible cases}} = \frac{c(n)}{c(s)}$$
3. Relative frequency approach of computing probability of an event A in $n$ trials of an experiment
   $$P(A) = \lim_{n\to\infty} \frac{c(A)}{n}$$
4. Rule of addition of two events
   - When events A and B are mutually exclusive
     P(A or B) = P(A) + P(B)
   - When events A and B are not mutually exclusive
     P(A or B) = P(A) + P(B) – P(A and B)
5. Conditional probability
   - For statistically independent events
     P(A|B) = P(A); P(B|A) = P(B)
   - For statistically dependent events
     $$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$
     $$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$
6. Rule of multiplication of two events
   - Joint probability of independent events
     P(A and B) = P(A)×P(B)
   - Joint probability of dependent events
     P(A and B) = P(A|B)×P(B)
     P(A and B) = P(B|A) × P(A)
7. Rule of elimination
   (i) $P(B) = \Sigma P(A_i)\, P(B|A_i)$
   (ii) $P(A) = \Sigma P(B_i)\, P(A|B_i)$
8. Baye's rule
   $$P(A_i|B) = \frac{P(A_i)\,P(B|A_i)}{\Sigma P(A_i)\,P(B|A_i)}$$
9. Basic rules for assigning probabilities
   - The probability assigned to each experimental outcome
     $0 \le P(A_i) \le 1$ for all $i$
   - Sum of the probabilities for all the experimental outcomes
     $\Sigma P(A_i) = P(A_1) + P(A_2) + \ldots + P(A_n) = 1$

   Complement of an event, $P(A) = 1 - P(\bar{A})$

# Chapter Concepts Quiz

## True or False

1. The classical approach to probability theory requires that the total number of possible outcomes be known or calculated and that each of the outcomes be equally likely. (T/F)
2. For any two statistically independent events, P(A or B) = P(A) + P(B). (T/F)
3. The marginal probability of an event can be formed by all the possible joint probabilities which include the event as one of the events. (T/F)
4. The posterior probabilities help a decision-maker to update his prior probabilities by using additional experimental data. (T/F)
5. The posterior probabilities are valid only when there are two elementary events and consistent outcomes. (T/F)
6. When someone says that the probability of occurrence of an event is 30 per cent, he is stating a classical probability. (T/F)
7. If $x = 4!/0!$, then $x$ is not well defined. (T/F)
8. Two events are mutually exclusive if their probabilities are less than one. (T/F)
9. If events are mutually exclusive and collectively exhaustive, then posterior probabilities for these events can be equal to their prior probabilities. (T/F)
10. *A priori* probability is estimated prior to receiving new information. (T/F)
11. An event is one or more of the possible outcomes which result from conducting an experiment. (T/F)
12. The condition of statistical independence arise when the occurrence of one event has no effect upon the probability of occurrence of any other event. (T/F)
13. The collective exhaustive list of outcomes to an experiment contains every single outcome possible. (T/F)
14. A marginal probability is also known as unconditional probability. (T/F)
15. The relative frequency approach of assessing the probability of some event gives the greatest flexibility. (T/F)

## Multiple Choice

16. If events are mutually exclusive, then
    (a) their probabilities are less than one
    (b) their probabilities sum to one
    (c) both events cannot occur at the same time
    (d) both of them contain every possible outcome of an experiment
17. Posterior probabilities for certain events are equal to their prior probabilities provided:
    (a) all the prior probabilities are zero
    (b) events are mutually exclusive and collectively exhaustive
    (c) events are statistically independent
    (d) none of the above
18. Two events A and B are statistically independent when
    (a) $P(A \cap B) = P(A) \times P(B)$
    (b) $P(A|B) = P(A)$
    (c) $P(A \cup B) = P(A) + P(B)$
    (d) both (a) and (b)
19. If $P(A \cap B) = A(A|B) \times P(B)$, then it implies that:
    (a) both events are statistically dependent and independent
    (b) both events are statistically dependent
    (c) both events are statistically independent
    (d) none of the above
20. What is the probability that a value chosen at random from a population is larger than the median of the population?
    (a) 0.25 (b) 0.50
    (c) 0.75 (d) none of the above
21. Baye's Theorem is useful in:
    (a) revising probability estimates
    (b) computing conditional probabilities
    (c) computing sequential probabilities
    (d) none of the above
22. A probability of getting the digit 2 in a throw of unbiased dice is:
    (a) zero (b) 1/2
    (c) 1/6 (d) 3/4
23. If two dice are thrown simultaneously, then the probability of getting a total of 6 will be:
    (a) 1/36 (b) 3/36
    (c) 5/36 (d) 7/36
24. A bag contains 3 red, 6 white, and 7 blue balls. If two balls are drawn at random, then the probability of getting both white balls is:
    (a) 5/40 (b) 6/40
    (c) 7/40 (d) 14/40
25. What is the probability of getting an odd number in tossing a dice?
    (a) 1/6 (b) 1/3
    (c) 1/2 (d) 1
26. What is the probability of getting more than 4 in rolling a dice?
    (a) 1/6 (b) 1/3
    (c) 1/2 (d) 1
27. If the outcome is an odd number when a dice is rolled, then the probability that it is a prime number:
    (a) 1/3 (b) 2/3
    (c) 1/6 (d) 5/6
28. If $P(A \cap B) = 0.20$ and $P() = 0.80$, then $P(A \mid B)$ is
    (a) 0.25 (b) 0.40
    (c) 0.50 (d) 0.75
29. If $P(AB) = 0$, then the events A and B are
    (a) independent (b) dependent
    (c) equally likely (d) none of these
30. If $P(A \cap B) = 0.60$ and $P(A \cup B) = 0.70$ for two events A and B, then $P(A) + P(B)$ is
    (a) 0.10 (b) 0.90
    (c) 1.00 (d) 0.75
31. If one event is unaffected by the outcome of another event, the two events are said to be
    (a) dependent (b) independent
    (c) mutually exclusive (d) joint
32. If $P(A \cup B) = P(A)$, then events A and B are
    (a) mutually exclusive (b) dependent
    (c) independent (d) none of these
33. If probability of choosing a value at random from a particular population is larger, then the median of population is
    (a) 0.25 (b) 0.50
    (c) 0.75 (d) 1.00
34. The chance of rain to day is 80 per cent. Which of the following best explain this statement
    (a) It will rain 80 per cent today
    (b) It will rain in 80 per cent of the area to which this forecast applies today
    (c) In the past, weather conditions of this short have produced rain in this area 80 per cent of the time.
    (d) none of these
35. Which of the following pairs of events are mutually exclusive?
    (a) A contractor losses a major contract, and he increases his work force by 50 per cent.
    (b) A man is older than his uncle and he is younger than his cousins.
    (c) A football team loses its last game of the year, and it wins the world cup.
    (d) none of these

### Concepts Quiz Answers

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1. T | 2. F | 3. T | 4. T | 5. F | 6. F | 7. F | 8. F | 9. T |
| 10. T | 11. T | 12. T | 13. T | 14. T | 15. F | 16. (c) | 17. (b) | 18. (d) |
| 19. (a) | 20. (b) | 21. (a) | 22. (c) | 23. (c) | 24. (a) | 25. (c) | 26. (b) | 27. (b) |
| 28. (c) | 29. (d) | 30. (a) | 31. (b) | 32. (d) | 33. (b) | 34. (c) | 35. (c) | |

## Review Self-Practice Problems

**6.42** Suppose a nationwide screening programme instituted through schools is being considered to uncover child abuse. It is estimated that 2 per cent of all children are subject to abuse. Further, existing screening programmers are able to determine correctly that abuse occurs 92 per cent of the time and that abuse is incorrectly suspected 5 per cent of the time.

(a) What is the probability that the results of screening indicating abuse are associated with children who are actually not abused?

(b) Based upon every 1,00,000 children screened, how many screenings can be expected to lead to a false accusation of abuse?

(c) Based upon your answer to part (a), is it valid to conclude that 73 per cent of the families not abusing children would be falsely accused? Why or why not ?

[*Delhi Univ., MBA, 1998*]

**6.43** If there is an increase in capital investment next year, the probability that the price of structural steel will increase is 0.90. If there is no increase in such investment, the probability of an increase is 0.40. Overall, we estimate that there is a 60 per cent chance that capital investment will increase next year.

(a) What is the overall probability of an increase in structural steel prices next year?

(b) Suppose that during the next year structural steel prices in fact increase, what is the probability that there was an increase in capital investment?

[*Delhi Univ., MBA, 2000*]

**6.44** A product is assembled from three components $X$, $Y$, and $Z$, and the probability of these components being defective is 0.01, 0.02, and 0.05. What is the probability that the assembled product will not be defective?

[*Delhi Univ., MBA, 2001*]

**6.45** A human resource manager has found it useful to categorize engineering job applicants according to their degree in engineering and relevant work experience. Out of all applicants for the job 70 per cent have a degree with or without any work experience, and 60 per cent have work experience with or without the degree. Fifty per cent of the applicants have both the degree and relevant work experience.

(a) Determine the probability that a randomly selected job applicant has either the degree or relevant work experience.

(b) What is the probability that the applicant has neither the degree nor work experience?

**6.46** A salesman is found to complete a sale with 10 per cent of potential customers contacted. If the salesman randomly selects two potential customers and calls on them, then (a) what is the probability that both the calls will result in sales? and (b) what is the probability that the two calls will result in exactly one sale?

**6.47** Suppose 80 per cent of the material received from a vendor is of exceptional quality, while only 50 per cent of the material received from vendor B is of exceptional quality. However, the manufacturing capacity of vendor A is limited, and for this reason only 40 per cent of the material purchased comes from vender A. The other 60 per cent comes from vendor B. An incoming shipment of material is inspected and it is found to be of exceptional quality. What is the probability that it came from vender A.

**6.48** The municipal corporation routinely conducts two independent inspections of each restaurant, with the restaurant passing only if both inspectors pass it. Inspector A is very experienced, and hence, passes only 2 per cent of restaurants that actually do have rules violations. Inspector B is less experienced and passes 7 per cent restaurants with violations. What is the probability that:

(a) A reports favourable, given that B has found a violation?

(b) B reports favourable with a violation, given that inspector A passes it?

(c) A restaurant with a violation is cleared by the corporation.

**6.49** If a hurricane forms in the Indian Ocean, there is a 76 per cent chance that it will strike the western coast of India. From data gathered over the past 50 years, it has been determined that the probability of a hurricane's occurring in this area in any given year is 0.85. What is the probability that a hurricane will occur in the eastern Indian Ocean and strike India this year?

**6.50** A departmental store has been the target of many shoplifters during the past month, but owing to increased security precautions, 250 shoplifters have been caught. Each shoplifter's sex is noted, also noted is whether he/she was a first-time or repeat offender. The data are summarized in the table below:

| *Sex* | *First-Time Offender* | *Repeat Offender* |
|---|---|---|
| Male | 60 | 70 |
| Female | 44 | 76 |

Assuming that an apprehended shoplifter is chosen at random, find:

(a) The probability that the shoplifter is male.

(b) The probability that the shoplifter is a first-time offender, given that the shoplifter is male.

(c) The probability that the shoplifter is female, given that the shoplifter is a repeat offender.

(d) The probability that the shoplifter is female, given that the shoplifter is a first time offender.

**6.51** A doctor has decided to prescribe two new drugs to 200 heart patients in the following manner: 50 get drug A, 50 get drug B, and 100 get both. Drug A reduces the probability of a heart attack by 35 per cent drug, B reduces the probability by 20 per cent, and the two drugs, when taken together, work independently. The 200 patients were chosen so that each has an 80 per cent chance of having a heart attack. If a randomly selected patient has a heart attack, what is the probability that the patient was given both drugs?

**6.52** The Deputy Commissioner of Police is trying to decide whether to schedule additional patrol units in two sensitive areas, A and B, in his district. He knows that on any given day during the past year, the probabilities of major crimes and minor crimes being committed in area A were 0.478 and 0.602, respectively, and that the corresponding probabilities in area B were 0.350 and 0.523. Assume that major and minor crimes occur independently of each other and likewise that crimes in the two areas are independent of each other.

(a) What is the probability that no crime of either type will be committed in the area A on a given day?
(b) What is the probability that a crime of either type will be committed in the area B on a given day?
(c) What is the probability that no crime of either type will be committed in either areas on a given day?

**6.53** The press-room supervisor for a daily newspaper is asked to find ways to print the paper closer to distribution time, thus giving the editorial staff more leeway for last-minute changes. He has the option of running the presses at 'normal' speed or at 110 per cent of normal—'fast' speed. He estimates that these will run at the higher speed 60 per cent of the time. The roll of paper (the newsprint 'web') is twice as likely to tear at the higher speed which would mean stopping the presses temporarily

(a) If the web on a randomly-selected printing run has a probability of 0.112 of tearing, what is the probability that the web will not tear at normal speed?
(b) If the probability of tearing at fast speed is 0.20, what is the probability that a randomly-selected torn web occurred at normal speed?

*[Delhi Univ., MBA, 1999]*

**6.54** The result of conducting an examination in two papers, A and B, for 20 candidates were recorded as under: 8 passed in paper A, 7 passed in paper B, 8 failed in both papers. If out of these candidates one is selected at random, find the probability that the candidate (a) passed in both A and B, (b) failed only in A, and (c) failed in A or B.

**6.55** When two dice are thrown $n$ number of times, the probability of getting at least one double six is greater than 99 per cent. What is the least numerical value of $n$.

*[CA, Nov., 1998]*

**6.56** It is known from past experience that a football team will play 40 per cent of its matches on artificial turf this season. It is also known that a football player's chances of incurring a knee injury are 50 per cent higher if he is playing an artificial turf instead of grass. Further, if a player's probability of knee injury on artificial turf is 0.42, what is the probability that (a) a randomly selected player incurs a knee injury, and (b) a randomly selected player with a knee injury, incurred the injury playing on grass?

**6.57** In a locality of 5000 people, 1200 are above 30 years of age and 3000 are females. Out of 1200 who were above 30 years of age, 200 are females. A person is chosen at random and you are told that the person is female. What is the probability that she is above 30 years of age? *[IGNOU, 1997; Delhi Univ., MBA, 1998, 2001]*

**6.58** Suppose 5 men out of 100 and 25 women out of 1000 are colour blind. A colour blind person is chosen at random. What is the probability of his being male (assuming that male and females are equal in proportion).

**6.59** An organization dealing with consumer products wants to introduce a new product in the market. Based on its past experience, it has a 65 per cent chance of being successful and 35 per cent of not being successful. In order to help the organization to make a decision on the new product, that is, whether to introduce or not, it decides to get additional information on consumer attitude towards the product. For this purpose, the organization decides on a survey. In the past when a product of this type was successful, surveys yielded favourable indication 85 per cent of the time, whereas unsuccessful products received favourable survey indications 30 per cent of the time. Determine the posterior probability of the product being successful given the survey information. *[IGNOU, 1999]*

**6.60** Police Head Quarter classified crime by age (in years) of the criminal and whether the crime is violent or non-violent. A total of 150 crimes were reported in the last month as shown in the table below:

| *Type of crime* | *Age (in years)* | | | |
|---|---|---|---|---|
| | *Under 20* | *20–40* | *Over 40* | *Total* |
| Violent | 27 | 41 | 14 | 82 |
| Non-violent | 12 | 34 | 22 | 68 |
| | 39 | 75 | 36 | 150 |

(a) What is the probability of selecting a case to analyze and finding the crime was committee by someone less than 40 years old.
(b) What is the probability of selecting a case that involved a violent crime or an offender less than 20 years old?
(c) If two crimes are selected for review, then what is the probability that both are violent crimes?

**6.61** With each purchase of a large pizza at a Pizza shop, the customer receives a coupon that can be scratched to see if a prize will be awarded. The odds of winning a free soft drink are 1 in 10, and the odds of winning a free large pizza are 1 in 50. You plan to eat lunch tomorrow at the shop. What is the probability.

(a) The you will win either a large pizza or a soft drink?
(b) That you will not win a prize?
(c) That you will not win a prize on three consecutive visits to the Pizza shop?
(d) That you will win at least one prize on one of your next three visits to the Pizza shop?

**6.62** The boxes of men's shirts were received from the factor. Box 1 contained 25 sport shirts and 15 dress shirts. Box 2 contained 30sport shirts and 10 dress shirts. One of the boxes was selected at random, and a shirt was chosen at random from that box to be inspected. The shirt was a sport shirt. Given this information, what is the probability that the sport shirt came from box 1?

**6.63** There are four people being considered for the position of chief executive officer of an Enterprises. Three of the applications are over 60 years of age. Two are female, of which only one is over 60. All four applications are either over 60 years of age or female. What is the probability that a candidate is over 60 and female?

**6.64** A pharmaceutical company through an advertisement in a magazine, estimates that 1 percent of the subscribers will buy products. They also estimate that 05 percent of nonsubscribers will buy the product and that there is one chance in 20 that a person is a subscriber.
(a) Find the probability that a randomly selected person will buy the products.
(b) If a person buys the products what is the probability he subscribes to the magazine?
(c) If a person does not buy the products what is the probability he subscribes to magazine?

## Hints and Answers

**6.42** (a) 0.727 (b) 6740

**6.43** R = rise in price of structural steel,
I = capital investment increasing.

(a) $P(R) = P(I \cap R) \cup P(\bar{I} \cap R)$
$= P(I)\,P(R|I) + P(\bar{I})\,P(R|\bar{I})$
$= 0.60 \times 0.90 + 0.40 \times 0.40 = 0.70$

(b) $$P(I|R) = \frac{P(I \cap R)}{P(R)} = \frac{P(I)\,P(R|I)}{P(I)\,P(R|I) + P(\bar{I})\,P(R|\bar{I})}$$
$$= \frac{0.60 \times 0.90}{0.60 \times 0.90 + 0.40 \times 0.40}$$
$$= \frac{0.54}{0.70} = 0.77$$

**6.44** P (product not defective)
$= P(\bar{X})\,P(Y)\,P(Z) + P(X)\,P(\bar{Y})\,P(Z) + P(X)\,P(Y)\,P(\bar{Z})$
$= 0.99 \times 0.02 \times 0.05 + 0.01 \times 0.98 \times 0.05 + 0.01 \times 0.02 \times 0.95$
$= 0.00099 + 0.00049 + 0.00019 = 0.00167$

**6.45** D = degree holders; W = with work experience
(a) $P(D \cup W) = P(D \text{ or } W)$
$= P(D) + P(W) - P(D \cap W)$
$= 0.70 + 0.60 - 0.50 = 0.80$
(b) $P(\bar{D} \cap \bar{W}) = 1.00 - P(D \cup W)$
$= 1.00 - 0.80 = 0.20$

**6.46** $S_1, S_2$ = calls resulted in sales on both the customers, respectively
(a) $P(S_1 \text{ and } S_2) = P(S_1 \cap S_2) = P(S_1)\,P(S_2)$
$= 0.10 \times 0.10 = 0.01$
(b) $P(S_1 \cup S_2) = P(S_1 \cap\ ) \cup P(\ \cap S_2)$
$= 0.10 \times 0.90 + 0.90 \times 0.10 = 0.18$

**6.47** A = material supplied by vendor A
E = material is of exceptional quality.
$$P(A|E) = \frac{P(A \cap E)}{P(E)} = \frac{P(A)\,P(E|A)}{P(A)\,P(E|A) + P(B)\,P(E|B)}$$
$$= \frac{0.40 \times 0.80}{0.40 \times 0.80 + 0.60 \times 0.50}$$
$$= \frac{0.32}{0.62} = 0.516$$

**6.48** (a) $P(A \mid \bar{B}) = P(A) = 0.02$
(b) $P(B \mid A) = P(B) = 0.07$
(c) $P(A \cap B) = P(A)\,P(B) = 0.02 \times 0.07 = 0.0014$

**6.49** Let H = hurricane forming over Indian Ocean;
W = hurricane hits western coast of India,
$P(H \cap W) = P(H)\,P(W \mid H) = 0.76 \times 0.85 = 0.646$

**6.50** M = shoplifter is male, W = shoplifter is female
F = shoplifter is first time offender,
R = shoplifter is repeat offender
(a) $P(M) = (60 + 70) \div 250 = 0.520$
(b) $P(F|M) = P(F \cap M)/P(M) = \frac{60}{250} \div \frac{130}{250} = 0.462$
(c) $P(W|R) = P(W \cap R)/P(R) = \frac{76}{250} \div \frac{146}{250} = 0.521$
(d) $P(W|F) = P(W \cap F)/P(F) = \frac{44}{250} \div \frac{104}{250} = 0.423$

**6.51** H = heart attack; D = drug given

| *Drug* | *P(D)* | *P(H \| D)* |
|---|---|---|
| A | 50/200 = 0.25 | 0.80 × 0.65 = 0.520 |
| B | 50/200 = 0.25 | 0.80 × 0.80 = 0.640 |
| A and B | 100/200 = 0.50 | 0.80 × 0.65 × 0.80 = 0.416 |

| *P(H ∩ D)* | *P(D\|H) = P(H ∩ D)/P(H)* |
|---|---|
| 0.130 | 0.130/0.498 = 0.2610 |
| 0.160 | 0.160/0.498 = 0.3213 |
| 0.208 | 0.208/0.498 = 0.4177 |
| P(H) = 0.498 | |

$P[(A \text{ and } B)/H] = P[(A \text{ and } B) \cap H]/P(H) = 0.208/0.498 = 0.417$

**6.52** $M_1, M_2$ = major crime in district A and B, respectively
$m_1, m_2$ = minor crime in district A and B, respectively.
(a) $P(M_1 \cup m_1) = P(M_1) + P(m_1) - P(M_1 \cap m_1)$
$= P(M_1) + P(m_1) - P(M_1)\,P(m_1)$
$= 0.478 + 0.602 - 0.478 \times 0.602$
$= 0.792$
$\therefore\ P(\bar{M}_1 \cap \bar{m}_1) = 1 - 0.792 = 0.208$
(b) $P(M_2 \cup m_2) = P(M_2) + P(m_2) - P(M_2)\,P(m_2)$
$= 0.350 + 0.523 - 0.350 \times 0.523$
$= 0.690$
(c) P (crime in A) = 0.792; P (crime in B) = 0.690
P (no crime in A and B)
= 1 – P (crime in at least A or B)

$= 1 - [P(A) + P(B) - P(A \text{ and } B)]$
$= 1 - [P(A) + P(B) - P(A)\,P(B)] = 0.064$

**6.53** (a) Let $x$ = P (no tear given normal speed). Then
P (tear) = P (tear | normal speed) P(normal speed) + P(tear | fast speed) P(fast speed)
$0.112 = (1 - x)(0.4) + 2(1 - x)(0.6) = 1.60 - 1.60x$
$1.6x = 1.6 - 0.112 = 1.488$ or $x = 1.488/1.6 = 0.93$

(b)

| Speed | Prob. | P (tour\|speed) | P (tear and speed) | P (speed\|tear) |
|---|---|---|---|---|
| Normal | 0.40 | 0.10 | 0.04 | 0.04/0.16 = 0.25 |
| Fast | 0.60 | 0.20 | 0.12 | 0.12/0.16 = 0.75 |
| | | P (tear) = | 0.16 | |

P (normal speed | tear) = 0.25

**6.54** (a) $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$$\frac{8}{20} + \frac{7}{20} - \left(1 - \frac{8}{20}\right) = \frac{3}{20}$$

(b) $P(\bar{A} \cap B) = P(\bar{A}) \times P(B) = \left(\frac{12}{20}\right)\left(\frac{7}{20}\right) = 0.21$

(c) $P(\bar{A} \cup \bar{B}) = A(\bar{A}) + P(\bar{B}) - (\bar{A} \cap \bar{B})$

$$= \frac{12}{20} + \frac{13}{20} - \frac{8}{20} = \frac{17}{20}$$

**6.55** Given $1 - (35/36)^n > 0.99$ or $n = 164$.

**6.56** A = knee injury; B = playing on artificial turf; C = playing on grass

(a) $P(A \cap B) = P(B) \cdot P(A \mid B) = 0.40 \times 0.42 = 0.168$
$P(A \cap C) = P(C)\,P(A \mid C) = 0.60 \times 0.28 = 0.168$
Thus $P(A) = P(A \cap B) + P(A \cap C)$
$= 0.168 + 0.168 = 0.336$

(b)
$$P(C \mid A) = \frac{P(A \cap C)}{P(A)}$$
$$= \frac{P(C) \cdot P(A|C)}{P(C)\,P(A|C) + P(B)\,P(A|B)}$$
$$= \frac{0.168}{0.168 + 0.168} = \frac{1}{2}$$

**6.57** P(A) = probability that a person chosen is above 30 years = 1200/5000 = 0.214
P(B) = probability that a person chosen is female = 3000/5000 = 0.60
P(A and B) = $P(A \cap B)$ = probability that a person chosen is above 30 years and a female
= 200/5000 = 0.04

$$\text{But } P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.04}{0.06} = \frac{1}{15}$$

**6.58** Let M, F and C denote male, female and colour blind persons. Then
P(M | C) = 5/100 = 1/20;
P (F | C) = 25/1000 = 1/40;
P (M) = P (F) = 1/2.

$$\text{Thus } P(C \mid M) = \frac{P(C \cap M)}{P(M)}$$
$$= \frac{P(M)\,P(M|C)}{P(M)\,P(M|C) + P(F)\,P(F|C)}$$
$$= \frac{(1/2)(1/20)}{(1/2)(1/20) + (1/2)(1/40)} = \frac{2}{3}$$

**6.59** $A_1, A_2$ = new product is successful and failure, respectively
I = additional information

| Event | Probability (1) | Conditional Probability $P(I \mid A_i)$ (2) | Joint Probability $P(A_i \cap I)$ (3) = (1) × (2) | Posterior Probability $P(A_i \mid I)$ (4) = (3)/(1) |
|---|---|---|---|---|
| $A_1$ | 0.65 | 0.85 | 0.552 | 0.84 |
| $A_2$ | 0.35 | 0.30 | 0.105 | 0.16 |

Posterior probability of product being successful given the survey information is 0.84.

**6.60** (a) P(crimes both violent and non-violent) committed by a person less than 40 years old)

$$= \frac{39}{150} + \frac{75}{100} = \frac{114}{150} = 0.76$$

(b) P(crime of violent type or offender less than 20 years old)

$$= \frac{82}{150} + \frac{39}{150} - \frac{27}{150} = \frac{94}{150} = 0.6267$$

(c) P(both crimes are of violent nature)

$$\frac{82}{150} \times \frac{81}{149} = \frac{6642}{22,300} = 0.2972$$

**6.61** Let A and B represent the event of winning pizza and soft drink, respectively.

(a) $P(A \text{ or } B) = P(A)\,P(\bar{B}) + P(\bar{A})\,P(B)$

$$= \frac{1}{50} \times \frac{9}{10} + \frac{49}{50} \times \frac{1}{10} = \frac{9}{500} + \frac{49}{500} = \frac{58}{500} = 0.116$$

(b) P(no prize) $= [1 - P(A)]\,[1 - P(B)] = P(\bar{A})\,P(\bar{B})$

$$= \frac{49}{500} \times \frac{9}{10} = \frac{441}{500} = 0.882$$

(c) P(no prize on 3 visits) = $[P(\text{no prize})]^3 = (0.882)^2$
= 0.686

(d) P(at least are prize) = 1 – P(no prize) = 1 – 0.686
= 0.314

**6.62** Given P(sport shirt) = 25/40, P(dress shirt) = 15/40 (in Box 1); P(sport short) = 30/40 P(dress shirt) = 10/40
P (shirt came from Box 1/shirt was short-shirt)

$$= \frac{P(\text{Box 1 and sport shirt})}{P(\text{sport shirt})}$$
$$= \frac{P(\text{sport shirt/Box 1})\,P(\text{Box 1})}{P(\text{Box 1})\,P(\text{sport/Box 1}) + P(\text{Box 2})\,P(\text{sport shirt/Box 2})}$$
$$= \frac{(25/40)(1/2)}{(25/40)(1/2) + (30/40)(1/2)}$$
$$= \frac{0.625 \times 0.50}{0.625 \times 0.50 + 0.75 \times 0.50} = \frac{0.3125}{0.6875} = 0.4545$$

**6.63** P(F and over 60 years) = P(F) × P(over 60 years)

$$= \frac{2}{4} \times \frac{1}{2} = 0.25$$

**6.64** Given P(subscribers buy product) = 0.05;
P(Non-subscribers buy product) = 0.95
P(buy | S) = 0.01, P(not buy | S) = 0.99;
P(buy | NS) = 0.005, P(not buy | NS) = 0.995

(a) P(buy) = P(S) P(buy | S) + P(NS) P(buy | NS)
= 0.05 × 0.01 + 0.95 × 0.005 = 0.00525

(b) P(S | buy) = (0.05 × 0.01)/0.00525 = 0.0952

(c) P(S | not buy) = (0.05 × 0.99)/0.00525 = 0.0498

# Case Studies

## Case 6.1: Tiger Air Express*

Tiger Air Express Company was founded in 1987, primarily to provide charter air services. Soon after, it incorporated tour business and taxi business into its domain and during its first two years of operations, its sales grew by 168 per cent.

Because of its rapid growth, it experienced some difficulties in providing the necessary resources and building proper infrastructure for this growth. These difficulties were due to underestimated requirement of capital for vehicles, maintenance, office facilities, and staff. The Gulf War which resulted in steep increase in fuel costs added more burden on the financially-strapped organization.

The management was having difficulty in exactly evaluating the financial and operational situation of the company. A well-known accounting firm was hired to develop and install an accounting system that would accurately reflect the Company's financial situation at any given time. As a result of the report, the taxi division was eliminated and the resources were diverted towards the freight and tour divisions of the company, specially the air-freight capabilities.

In order to expand on the air-freight business, further studies were conducted in order to understand the market better. Some insights into the air-freight market were gained through a survey of 44 air freight shippers conducted by an outside agency in 1992. There are two types of services available for forwarding air freight. First, there are integrated carriers. These are the companies which have their own planes. An example is Federal Express. These carriers offer customers pickup and delivery by truck along with airplane shipping. Second, there are freight forwarders. These are the carriers that handle pickup and delivery but use the regular scheduled airlines for air shipment.

A summary of the responses to some of the questions asked in the survey, as noted above, are explained below:

(i) 30 per cent of the survey respondents stated that their air-freight expenditures had increased in the past year. 60 per cent indicated no change in these expenditures and 10 per cent reported a drop in expenses.

(ii) 50 per cent of the survey respondents used integrated carriers for shipping. However, for small packages, this percentage increased to 75 per cent. Only 19 per cent used freight forwarders. 62 per cent of those who used freight forwarders did so because of their ability to handle international air freight more efficiently and reliably.

(iii) 39 per cent of the respondents indicated that they were using more 2-3 days deliveries which was cheaper than one day delivery, than they did 2 years ago.

(iv) When the survey asked the customers as to what they looked for in a carrier the responses were as follows:

- On-time delivery : 59 per cent
- Price : 57 per cent
- Customer responsiveness : 43 per cent
- Geographic coverage : 9 per cent
- Single source control : 7 per cent
- Tracing capabilities : 7 per cent.

### Questions for Discussion

1. Based on your knowledge of probability theory, what strategy should Tiger Air Express adopt in determining their emphasis on the air freight business, based on the data given?
2. Some of the percentages reported in the survey can be converted into conditional probabilities. Conditional probabilities contain information about the responses in certain categories. Can you describe some of these categories? How would these categories help the management of Tiger Air Express? Explain.
3. If you were a consultant to Tiger Air Express, what recommendations would you give to the management so that they can meet the competition with other companies, offerings based on the type of shippers and shipments that are most probably in demand.

* Adapted from J S Chandan 'Statistics for Business and Economics' *Tiger Air Express Inc.*; Warner Books, 1991, '*What Do Air Shippers Want? Aircargo Survey*', *Traffic Management*, July 1992.

*Any body can win unless there happens to be a second entry*

—George Alda

*When it is not in our power to determine what is true, we ought to follow what is most probable.*

—Descartes

Chapter 7

# Probability Distributions

## LEARNING OBJECTIVES

After studying this chapter, you should be able to

- define the terms random variable and probability distribution.
- distinguish between discrete and continuous probability distributions.
- describe the characteristics and compute probabilities using both discrete and continuous probability distributions.
- compute expected value and variance of a random variable.
- apply the concepts of probability distributions to real-life problems.

## 7.1 INTRODUCTION

In any probabilistic situation each strategy (course of action) may lead to a number of different possible outcomes. For example, a product whose sale is estimated around 100 units, may be equal to 100, less, or more. Here the sale (i.e., an outcome) of the product is measured in real numbers but the volume of the sales is uncertain. The volume of sale which is an uncertain quantity and whose definite value is determined by chance is termed as *random (chance* or *stochastic) variable.* A listing of all the possible outcomes of a random variable with each outcome's associated probability of occurrence is called *probability distribution*. The numerical value of a random variable depends upon the outcome of an experiment and may be different for different trials of the same experiment. The set of all such values so obtained is called the *range space* of the random variable.

In all such cases as mentioned above, the decision-maker may like to know

(*i*) the average value (payoff) of the random variable, and
(*ii*) the risk involved in choosing a strategy.

**Illustration:** If a coin is tossed twice, then the sample space of events, for this random experiment is

$$S = \{H\,H, T\,H, H\,T, T\,T\}$$

In this case, if the decision-maker is interested to know the probability distribution for the number of heads on two tosses of the coin, then a random variable ($x$) may be defined as:

$$x = \text{number of H's occurred}$$

The values of $x$ will depend on chance and may take the values: H H = 2, H T = 1, T H = 1, T T = 0. Thus the range space of $x$ is {0, 1, 2}.

When a random variable $x$ is defined, a value is given to each simple event in the sample space. The probability of any particular value of $x$ can then be found by adding the probabilities for all the simple events that have that value of $x$. For example, the probabilities of occurrence of 'heads' can be associated with each of the random variable values. Supposing $P(x = r)$ represents the probability of the random variable taking the value $r$ (here $r$ represents the number of heads occurred). Then probabilities of occurrence of different number of heads are computed as:

| *Number of Heads (x)* | *Probability of Outcome P(x)* |
|---|---|
| 0 | $P(x = 0) = P(T\,T) = P(T) \times P(T) = 0.5 \times 0.5 = 0.25$ |
| 1 | $P(x = 1) = P(H\,T) + P(T\,H) = P(H) \times P(T) + P(T) \times P(H)$ |
| | $= 0.5 \times 0.5 + 0.5 \times 0.5 = 0.25 + 0.25 = 0.50$ |
| 2 | $P(x = 2) = P(H\,H) = P(H) \times P(H) = 0.5 \times 0.5 = 0.25$ |

**Discrete random variable:** A variable that is allowed to take on only integer values.

**Broad Classes of Random Variable** A random variable may be either discrete or continuous. A **discrete random variable** can take on only a finite or countably infinite number of distinct values such as 0, 1, 2, . . .,. A discrete random variable is usually the result of counting. The number of letters received by a post office during a particular time period, the number of machines breaking down on a given day, the number of vehicles arriving at a toll bridge, and so on, are a few examples of discrete random variables.

**Continuous random variable:** A variable that is allowed to take on any value within a given range

A **continuous random variable** can take any numerical value in an interval or collection of intervals. A continuous random variable is usually the result of experimental outcomes that are based on measurement scales. For instance, measurement of time, weight, distance, temperature, and so on are all treated as continuous random variables. Tonnage produced by a steel blast furnace, amount of rainfall in a rainy season, height of individuals, time between arrival of customers at a service system in minutes, are also few examples of continuous random variables.

## 7.2 PROBABILITY DISTRIBUTION FUNCTION (*pdf*)

Probability distribution functions can be classified into two categories:

- *Discrete* probability distributions
- *Continuous* probability distributions

**Discrete probability distribution:** A probability distribution in which the random variable is permitted to take on only integer values.

A **discrete probability distribution** assumes that the outcomes of a random variable under study can take on *only integer values*, such as:

- A book shop has only 0, 1, 2, ... copies of a particular title of a book
- A consumer can buy 0, 1, 2, ... shirts, pants, etc.

If the random variable $x$ is discrete, its probability distribution called *probability mass function (pmf)* must satisfy following two conditions:

(i) The probability of a any specific outcome for a discrete random variable must be between 0 and 1. Stated mathematically, $0 \leq f(x = k) \leq 1$, for all value of $k$

(ii) The sum of the probabilities over all possible values of a discrete random variable must equal 1. Stated mathematically, $\sum_{\text{all } k} f(x = k) = 1$

A continuous probability distribution assumes that the outcomes of a random variable can take on only value in an interval such as:

- Product costs and prices.
- Floor area of a house, office, etc.

If the random variable $x$ is continuous, then its probability density function must satisfy following two conditions:

(i) $P(x) \geq 0$ ; $-\infty < x < \infty$ (non-negativity condition)

(ii) $\int_{-\infty}^{\infty} P(x)\,dx = 1$ (Area under the continuous curve must total 1)

**Continuous probability distribution:** A probability distribution in which the random variable is permitted to take any value within a given range

**Continuous probability distribution** functions are used to find probabilities associated with random variable values $x_1, x_2, \ldots, x_n$ in a given interval or range, say $(a, b)$. In other words, these probabilities are determined by finding the area under the *pdf* between the values $a$ and $b$. Mathematically, the area under *pdf* between $a$ and $b$ is given by

$$f(a \leq x \leq b) = f(b) - f(a) = \int_a^b f(x)\,dx$$

We can express $f(a \leq x \leq b)$ in terms of a distribution function, $f(x)$, provided it is differentiable. That is,

$$\frac{d}{dx}\{f(x)\} = \frac{d}{dx}\left\{\int_a^b f(x)\,dx\right\}$$

**Figure 7.1**
Probability Distribution Function

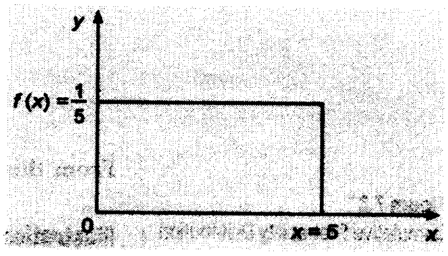



**Illustration:** Consider the function, $f(x) = \begin{cases} a & 0 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$

For $f(x)$ to be a *pdf*, the condition, $\int_{-\infty}^{\infty} f(x)\,dx = 1$ must be satisfied, which is true if

$$\int_0^5 a\,dx = 1, \text{ i.e. } a = \frac{1}{5}.$$

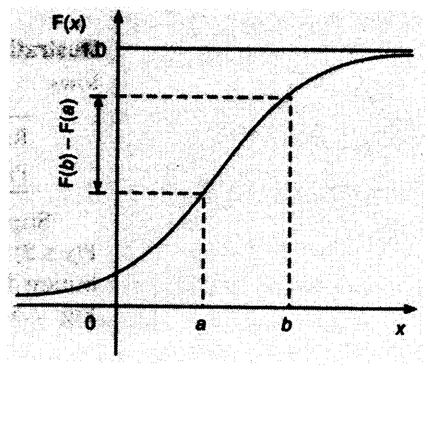
Since $a > 0$, the function, $f(x) \geq 0$. Thus $f(x)$ satisfies both the conditions for a *pdf*. Figure 7.1 illustrates the function graphically

## 7.3 CUMULATIVE PROBABILITY DISTRIBUTION FUNCTION (*cdf*)

The cumulative probability distribution function for the continuous random variable $x$ $(-\infty < x < \infty)$ is a rule or table that provides the probabilities $P(x \leq k)$ for any real number $k$. Generally, the term cumulative probability refers to the probability that $x$ is less than or equal to a particular value. For example, if we have three values of a random variable $x$ as: $a < b < c$, then

$$\int_a^c f(x) \geq \int_a^b f(x)$$

**Figure 7.2**
Cumulative Probability Distribution Function




This condition shows that *cdf* increases from left to right as shown in Fig. 7.2. Thus the probability that the value of the random variable $x$ is less than any real number $a$, is given by

$$F(a) = P(x \leq a) = \int_{-\infty}^{a} f(x)\,dx$$

where the function $F(a)$, also called the cumulative distribution (or function), represents the probability that $x$ does not exceed a specified value '$a$', and the area under the $f(x)$ curve to the left of the value $a$. That is, the probability of the random variable $x$ lies at or below some specific value, $a$. The *cdf* has the properties

(i) F($a$) is non-decreasing function
(ii) F($-\infty$) = 0 and F($\infty$) = 1.

In general, given two real numbers $a$ and $b$ such that $a < b$, the probability that the value of $x$ lies in any specified range, say between $a$ and $b$ is,

$$P(a \le x \le b) = \int_a^b f(x)\,dx = \int_{-\infty}^b f(x)\,dx - \int_{-\infty}^a f(x)\,dx = F(b) - F(a)$$

A typical *cdf* is illustrated in Fig. 7.2. However, if P($x = a$) and P($x = b$), then both of these have zero value in a continuous distribution. That is

$$P(x = a) = \int_a^a f(x) = 0$$

This result does not mean that the value of the random variable cannot be exactly equal to $a$. There is an infinitely large number of possible values and the probability associated with any one of them is zero. Thus *cdf* has the following properties:

$$\lim_{a \to \infty} F(a) = \lim_{a \to \infty} \int_{-\infty}^a f(x)\,dx = 1$$

$$\lim_{a \to -\infty} F(a) = \lim_{a \to -\infty} \int_{-\infty}^a f(x)\,dx = 0$$

From this relationship it follows that, $f(x) = \frac{d}{dx} F(x)$.

**Figure 7.3**
Cumulative Probability Distribution Function

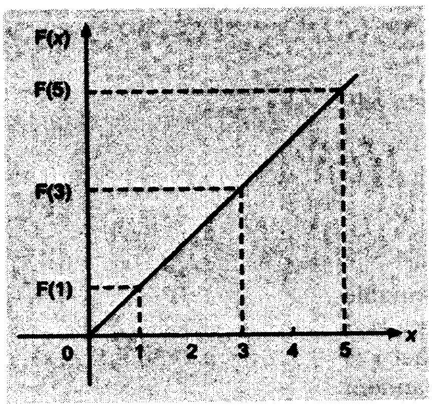



**Illustration:** For the continuous *pdf* is defined as

$$f(x) = \begin{cases} 1/5, & 0 \le x \le 5 \\ 0, & \text{otherwise} \end{cases}$$

the *cdf* in the range $0 \le x \le 5$ is given by

$$F(x) = \int_0^x f(x)\,dx = \int_0^x \frac{1}{5}\,dx = \frac{x}{5}$$

The *cdf* is illustrated in Fig 7.3.
For a given *pdf*, suppose we want to calculate $P(1 \le x \le 3)$. Then

$$P(1 \le x \le 3) = F(3) - F(1) = \frac{3}{5} - \frac{1}{5} = \frac{2}{5}$$

This may be seen from Fig. 7.3.

The cumulative probability distribution function of a discrete variable also specifies the probability that an observed value of discrete random variable $x$ will be no greater than a value, say $k$. In other words, if F($k$) is a *cdf* and $f(k)$ is a *pmf*, then

$$F(k) = P(x \le k) = f(x \le k)$$

**Illustration:** Let the probability distribution function of the discrete variable $x$ be as follows:

| Random variable : | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Probability, P($x = a$) : | 0.10 | 0.20 | 0.40 | 0.20 | 0.10 |

Suppose we want to know the probability of $x$ being equal to or less than 2, that is, P($x \le 2$) = 0.70. The probability distribution function and cumulative probability distribution function of 'less than or equal to' type are shown in Table 7.1 and graphed in Fig. 7.4.

**Table 7.1: Cumulative Distribution Function**

| | | |
|---|---|---|
| 0 | 0.10 | 0.10 |
| 1 | 0.20 | 0.30 |
| 2 | 0.40 | 0.70 |
| 3 | 0.20 | 0.90 |
| 4 | 0.10 | 1.00 |

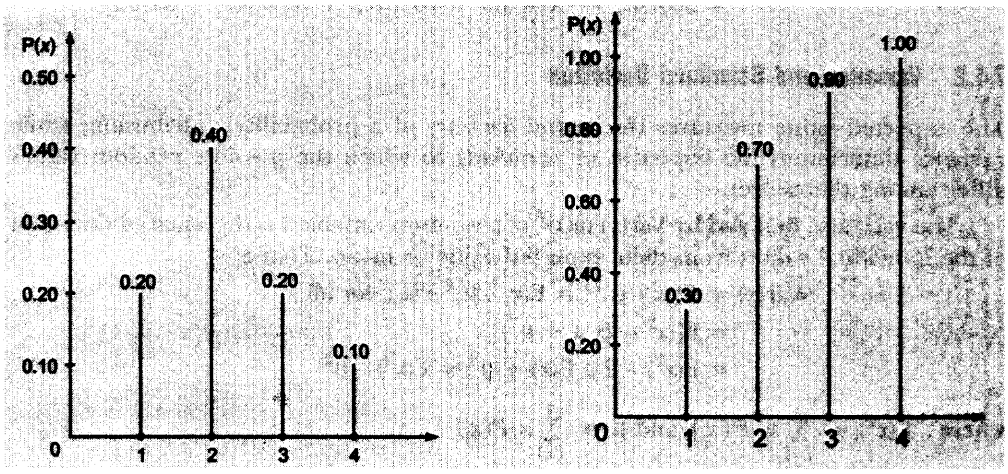



**Figure 7.4 (a)** Probability Distribution Function **Figure 7.4 (b)** Cumulative Distribution Function

For example, if we want to know the probability of $x$ being greater than 2, then it is given by

$$P(x > 2) = 1 - P(x \leq 2) = 1 - 0.70 = 0.30$$

Thus given the probability distribution function, we can obtain the cumulative distribution function.

## 7.4 EXPECTED VALUE AND VARIANCE OF A RANDOM VARIABLE

In the same way as discussed in Chapter 3 and 4, a probability distribution is also summarized by its mean and variances.

### 7.4.1 Expected Value

The mean (also referred as **expected value) of a random variable** is a typical value used to summarize a probability distribution. It is the weighted average, where the possible values of random variable are weighted by the corresponding probabilities of occurrence. If $x$ is a random variable with possible values $x_1, x_2, \ldots, x_n$ occuring with probabilities $P(x_1), P(x_2), \ldots, P(x_n)$, then the expected value of $x$ denoted by $E(x)$ or $\mu$ is the sum of the values of the random variable weighted by the probability that the random variable takes on that value.

**Expected value of a random variable:** A weighted average obtained by multiplying each possible value of the random variable with its probability of occurrence.

$$E(x) = \sum_{j=1}^{n} x_j P(x_j), \text{ provided } \sum_{j=1}^{n} P(x_j) = 1$$

Similarly, for the continuous random variable, the expected value is given by:

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

where $f(x)$ is the probability distribution function.

If E(x) is calculated in terms of rupees, then it is known as *expected monetary value* (EMV). For example, consider the price range of an item along with the probabilities as below:

| | | | | | |
|---|---|---|---|---|---|
| Price, $x$ | : | 50 | 60 | 70 | 80 |
| Probability, P(x) | : | 0.2 | 0.5 | 0.2 | 0.1 |

Thus the expected monetary value of the item is given by

$$\begin{aligned} \text{EMV}(x) &= \sum_{j=1}^{n} x_j P(x_j) \\ &= 50 \times 0.2 + 60 \times 0.5 + 70 \times 0.2 + 80 \times 0.1 = \text{Rs } 62. \end{aligned}$$

### 7.4.2 Variance and Standard Deviation

The expected value measures the *central tendency* of a probability distribution, while variance determines the *dispersion* or *variability* to which the possible random values differ among themselves.

The variance, denoted by Var(x) or $\sigma^2$ of a random variable $x$ is the squared deviation of the individual values from their expected value or mean. That is

$$\begin{aligned} \text{Var}(x) &= E[(x-\mu)^2] = E(x_j - \mu)^2 \, P(x_j), \text{ for all } j \\ &= E[(x^2 - 2x\mu + \mu^2)] \\ &= E(x^2) - 2\mu E(x) + \mu^2 = E(x^2) - \mu^2 \end{aligned}$$

where $E(x^2) = \sum_{j=1}^{n} x_j^2 P(x_j)$ and $\mu = \sum_{j=1}^{n} x_j P(x_j)$

The variance has the disadvantage of squaring the unit of measurement. Thus, if a random variable is measured in rupees, the variance will be measured in rupee squared. This shortcoming can be avoided by using *standard deviation* ($\sigma_x$) as a measure of dispersion so as to have the same unit of measurement. That is

$$\sigma_x = \sqrt{\text{Var}(x)} = \sqrt{\sum_{j=1}^{n} (x_j - \mu)^2 P(x_j)}$$

### 7.4.3 Properties of Expected Value and Variance

The following are the important properties of an expected value of a random variable:

1. The expected value of a constant $c$ is constant. That is, $E(c) = c$, for every constant $c$.
2. The expected value of the product of a constant $c$ and a random variable $x$ is equal to constant $c$ times the expected value of the random variable. That is, $E(cx) = cE(x)$.
3. The expected value of a linear function of a random variable is same as the linear function of its expectation. That is, $E(a + bx) = a + bE(x)$.
4. The expected value of the product of two independent random variables is equal to the product of their individual expected values. That is, $E(xy) = E(x)\,E(y)$.
5. The expected value of the sum of the two independent random variables is equal to the sum of their individual expected values. That is, $E(x + y) = E(x) + E(y)$.
6. The variance of the product of a constant and a random variable X is equal to the constant squared times the variance of the random variable X. That is, $\text{Var}(cx) = c^2\,\text{Var}(x)$.
7. The variance of the sum (or difference) of two independent random variables equals the sum of their individual variances. That is, $\text{Var}(x \pm y) = \text{Var}(x) \pm \text{Var}(y)$.

**Example 7.1:** A doctor recommends a patient to take a particular diet for two weeks and there is equal chance for the patient to lose weight between 2 kgs and 4 kgs. What is the average amount the patient is expected to lose on this diet?

**Solution:** If $x$ is the random variable, then probability density function is defined as:

$$f(x) = \begin{cases} \frac{1}{2}, & 2 < x < 4 \\ 0, & \text{otherwise} \end{cases}$$

The amount the patient is expected to lose on the diet is:

$$E(x) = \int_2^4 x.\frac{1}{2}\,dx = \left[\frac{x^2}{4}\right]_2^4 = \frac{1}{4}\left[(4)^2 - (2)^2\right] = 3 \text{ kg}$$

**Example 7.2:** From a bag containing 3 red balls and 2 white balls, a man is to draw two balls at random without replacement. He gains Rs 20 for each red ball and Rs 10 for each white one. What is the expectation of his draw.

**Solution:** Let $x$ be the random variable denoting the number of red and white balls in a draw. Then $x$ can take up the following values.

$$P(x = 2 \text{ red balls}) = \frac{^3C_2}{^5C_2} = \frac{3}{10}$$

$$P(x = 1 \text{ red and } 1 \text{ white ball}) = \frac{^3C_1 \times {^2C_1}}{^5C_2} = \frac{3}{5}$$

$$P(x = 2 \text{ white balls}) = \frac{^2C_2}{^5C_2} = \frac{1}{10}$$

Thus, the probability distribution of $x$ is:

| | | | | |
|---|---|---|---|---|
| Variable | : | 2R | 1R and 1W | 2W |
| Gain, $x$ | : | 40 | 30 | 20 |
| Probability, $P(x)$: | | 3/10 | 3/5 | 1/10 |

Hence, expected gain is, $E(x) = 40\times(3/10) + 30\times(3/5) + 20\times(1/10)$
$= \text{Rs } 32.$

**Example 7.3:** Under an employment promotion programme, it is proposed to allow sale of newspapers inside buses during off-peak hours. The vendor can purchase newspapers at a special concessional rate of Rs 1.25 per copy against the selling price of Rs 1.50. Any unsold copies are, however, a dead loss. A vendor has estimated the following probability distribution for the number of copies demanded.

| Number of copies : | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|
| Probability : | 0.04 | 0.19 | 0.33 | 0.26 | 0.11 | 0.07 |

How many copies should be ordered so that his expected profit will be maximum?

**Solution:** Profit per copy = Selling price – Purchasing price = 1.50 – 1.25 = Re 0.25. Thus

Expected profit = Number of copies × Probability × Profit per copy

The calculations of expected profit are shown in the Table 7.2.

**Table 7.2: Calculations of Expected Profit**

| *Number of Copies Demanded* (1) | *Probability* (2) | *Profit per Copy (Rs)* (3) | *Expected Profit (Rs)* (4) = (1) × (2) × (3) |
|---|---|---|---|
| 15 | 0.04 | 0.25 | 15 |
| 16 | 0.19 | 0.25 | 76 |
| 17 | 0.33 | 0.25 | **140** |
| 18 | 0.26 | 0.25 | 117 |
| 19 | 0.11 | 0.25 | 52 |
| 20 | 0.07 | 0.25 | 35 |

The maximum profit of Rs 140 is obtained when he stocks 17 copies of the newspaper.

**Example 7.4:** In a cricket match played to benefit an ex-player, 10,000 tickets are to be sold at Rs 500. The prize is a Rs 12,000 fridge by lottery. If a person purchases two tickets, what is his expected gain?

**Solution:** The gain, say $x$ may take one of two values: he will either lose Rs. 1,000 (i.e. gain will be – Rs 1,000) or win Rs (12,000–1,000) = Rs 11,000, with probabilities 9,998/10,000 and 2/10,000, respectively. The probability distribution for the gain $x$ is given below:

| $x$ | $P(x)$ |
|---|---|
| – Rs 1000 | 9,998/10,000 |
| Rs 11000 | 2/10,000 |

The expected gain will be

$$\mu = E(x) = \Sigma x\ P(x)$$
$$= -1000 \times (9{,}998/10{,}000) + 11000 \times (2/10{,}000) = -\text{Rs } 997.6$$

The result implies that if the lottery were repeated an infinitely large number of times, average or expected loss will be Rs 997.6.

**Example 7.5:** A market researcher at a major automobile company classified households by car ownership. The relative frequencies of households for each category of ownership are shown below:

| *Number of cars Per House hold* | *Relative Frequency* |
|---|---|
| 0 | 0.10 |
| 1 | 0.30 |
| 2 | 0.40 |
| 3 | 0.12 |
| 4 | 0.06 |
| 5 | 0.02 |

Calculate the expected value and standard deviation of the random variable and interpret the result [*Delhi Univ., MBA, 2003*]

**Solution:** The necessary calculations required to calculate expected and standard deviation of a random variable, say $x$ are shown is Table 7.3.

**Table 7.3: Calculations of Expected Value and Standard Deviation**

| *Number of Cars Per Households x* | *Relative Frequency, P(x)* | $x \times P(x)$ | $x^2 \times P(x)$ |
|---|---|---|---|
| 0 | 0.10 | 0.10 | 0.00 |
| 1 | 0.30 | 0.30 | 0.30 |
| 2 | 0.40 | 0.80 | 1.60 |
| 3 | 0.12 | 0.36 | 1.08 |
| 4 | 0.06 | 0.24 | 0.96 |
| 5 | 0.02 | 0.10 | 0.50 |
| | | 1.80 | 4.44 |

Expected value, $\mu = E(x) = \Sigma x\ P(x) = 1.80$. This value indicates that there are on an average 1.8 cars per household

Variance, $\sigma^2 = \Sigma x^2\ P(x) - [E(x)]^2 = 4.44 - (1.80)^2 = 4.44 - 3.24 = 1.20$

Standard deviation $\sigma = \sqrt{\sigma^2} = \sqrt{1.20} = 1.095$ cars.

**Example 7.6:** The owner of a 'Pizza Hut' has experienced that he always sells between 12 and 15 of his famous brand 'Extra Large' pizzas per day. He prepares all of them in advance and store them in the refrigerator. Since the ingredients go bad within one day, unsold pizzas are thrown out at the end of each day. The cost of preparing each pizza is

Rs 120 and he sells each one for Rs 170. In addition to the usual cost, it cost him Rs. 50 per pizza that is ordered but can not be delivered due to insufficient stock. If following is the probability distribution of the number of pizzas ordered each day, then how many 'Extra Large' pizza should he stock each day in order to minimize expected loss.

| Number of pizza demanded : | 12 | 13 | 14 | 15 |
|---|---|---|---|---|
| Probability : | 0.40 | 0.30 | 0.20 | 0.10 |

**Solution:** The loss matrix for the given question is shown in Table 7.4

**Table 7.4: Pizza Ordered**

| | *Pizza Ordered* 12 | 13 | 14 | 15 | |
|---|---|---|---|---|---|
| *Probability →* *Pizza Stocked ↓* | 0.40 | 0.30 | 0.20 | 0.10 | *Expected Loss* |
| 12 | – | 100 | 200 | 300 | 100 |
| 13 | 120 | – | 100 | 200 | 88 |
| 14 | 240 | 120 | – | 100 | 142 |
| 15 | 360 | 240 | 120 | – | 240 |

Since expected loss of Rs 88 is minimum corresponding to a stock level of 13 pizza, the owner should stock 13 'Extra Large' pizzas each day.

**Example 7.7:** A company introduces a new product in the market and expects to make a profit of Rs 2.5 lakh during the first year if the demand is 'good'; Rs 1.5 lakh if the demand is 'moderate'; and a loss of Rs 1 lakh if the demand is 'poor.' Market research studies indicate that the probabilities for the demand to be good and moderate are 0.2 and 0.5 respectively. Find the company's expected profit and the standard deviation.

**Solution:** Let $x$ be the random variable representing profit in three types of demand. Thus, $x$ may assume the values:

$x_1$ = Rs 2.5 lakh when demand is good,
$x_2$ = Rs 1.5 lakh when demand is moderate, and
$x_3$ = Rs 1 lakh when demand is poor.

Since these events (demand pattern) are mutually exclusive and exhaustive, therefore

$$P(x_1)+P(x_2)+P(x_3) = 1 \text{ or } 0.2 + 0.5 + P(x_3) = 1 \text{ or } P(x_3) = 0.3$$

Hence, the expected profit is given by

$$E(x) = 2.5 \times 0.2 + 1.5 \times 0.5 + (-1) \times 0.3 = \text{Rs } 0.95 \text{ lakh}$$

Also $$E(x^2) = x_1^2\, P(x_1) + x_2^2\, P(x_2) + x_3^2\, P(x_3)$$
$$= (2.5)^2 \times 0.2 + (1.5)^2 \times 0.5 + (-1)^2 \times 0.3 = \text{Rs } 2.675 \text{ lakh}$$

Thus $$\text{S.D.}(x) = \sqrt{\text{Var}(x)} = \sqrt{E(x^2) - [E(x)]^2} = \sqrt{2.675 - (0.95)^2} = \text{Rs } 1.331 \text{ lakh.}$$

## Conceptual Questions 7A

1. Define 'random variable'. How do you distinguish between discrete and continuous random variables. Illustrate your answer with suitable examples.
2. (a) Define mathematical expectation of a random variable.
   (b) Explain what do you mean by the term 'mathematical expectation'. How is it useful for a businessman? Given an example to illustrate its usefulness. [*Delhi Univ., MBA, 1997*]
3. What is meant by probability distribution of a random variable? Distinguish between probability density function and probability mass function. Illustrate with examples.
4. What do you understand by the expected value of a random variable?
5. What are the properties of expected value and variance of a random variable?

## Self-Practice Problems 7A

**7.1** Anil company estimates the net profit on a new product it is launching to be Rs 30,00,000 during the first year if it is 'successful' Rs 10,00,000 if it is 'moderately successful'; and a loss of Rs 10,00,000 if it is 'unsuccessful'. The firm assigns the following probabilities to its first year prospects for the product: Successful : 0.15, moderately successful : 0.25. What are the expected value and standard deviation of first year net profit for this product?
[*Delhi Univ., MBA, 2003*]

**7.2** If the probability that the value of a certain stock will remain the same is 0.46, the probability that its value will increase by Re 0.50 or Re 1.00 per share are respectively 0.17 and 0.23, and the probability that its value will decrease Re 0.25 per share is 0.14, what is the expected gain per share?

**7.3** A box contains 12 items of which 3 are defective. A sample of 3 items is selected at random from this box. If $x$ represents the number of defective items of 3 selected items, describe the random variable $x$ completely and obtain its expectation.

**7.4** Fifty per cent of all automobile accidents lead to property damage of Rs 100, forty per cent lead to damage of Rs 500, and ten per cent lead to total loss, that is, damage of Rs 1800. If a car has a 5 per cent chance of being in an accident in a year, what is the expected value of the property damage due to that possible accident?

**7.5** The probability that there is atleast one error in an account statement prepared by A is 0.2 and for B and C it is 0.25 and 0.4 respectively. A, B, and C prepared 10, 16, and 20 statements respectively. Find the expected number of correct statements in all.

**7.6** A lottery sells 10,000 tickets at Re 1 per ticket, and the prize of Rs 5000 will be given to the winner of the first draw. Suppose you have bought a ticket, how much should you expect to win?

**7.7** The monthly demand for transistors is known to have the following probability distribution.

| Demand ($n$) : | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Probability ($P$) : | 0.10 | 0.15 | 0.20 | 0.25 | 0.18 | 0.12 |

Determine the expected demand for transistors. Also obtain the variance. Suppose the cost (C) of producing '$n$' transistors is given by the relationship, C = 10,000 + 500$n$. Determine the expected cost.

**7.8** A bakery has the following schedule of daily demand for cakes. Find the expected number of cakes demanded per day.

| *Number of Cakes Demanded* | *Probability* |
|---|---|
| 0 | 0.02 |
| 1 | 0.07 |
| 2 | 0.09 |
| 3 | 0.12 |
| 4 | 0.20 |
| 5 | 0.20 |
| 6 | 0.18 |
| 7 | 0.10 |
| 8 | 0.01 |
| 9 | 0.01 |

**7.9** A consignment of machine parts is offered to two firms, A and B, for Rs 75,000. The following table shows the probabilities at which firms A and B will be able to sell the consignment at different prices.

| *Probability* | *Price (in Rs) at which the Consignment Can be Sold* | | | |
|---|---|---|---|---|
| | 60,000 | 70,000 | 80,000 | 90,000 |
| A | 0.40 | 0.30 | 0.20 | 0.10 |
| B | 0.10 | 0.20 | 0.50 | 0.20 |

Which firm, A or B, will be more inclined towards this offer?

**7.10** An industrial salesman wants to know the average number of units he sells per sales call. He checks his past sales records and comes up with the following probabilities:

| Sales (units) : | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Probability : | 0.15 | 0.20 | 0.10 | 0.05 | 0.30 | 0.20 |

You are expected to help the salesman in his objective.

**7.11** A survey conducted over the last 25 years indicated that in 10 years the winter was mild, in 8 years it was cold, and in the remaining 7 it was very cold. A company sells 1000 woollen coats in a mild year, 1300 in a cold year, and 2000 in a very cold year. You are required to find the yearly expected profit of the company if a woollen coat costs Rs 173 and it is sold to stores for Rs 248.

## Hints and Answers

**7.1**

| $x$ : | 3 | 1 | –1 |
|---|---|---|---|
| P($x$) : | 0.15 | 0.25 | 1 – 0.15 – 0.25 = 0.60 |

E($x$) = Rs 0.10 million, Var ($x$) = Rs 2.19 million, and $\sigma_x$ = Rs 1.48 million.

**7.2** Rs. 0.28

**7.3**

| $x$ : | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| P($x$) : | 27/64 | 27/64 | 9/64 | 1/64; |

E($x$) = 0.75

**7.4** $x$ : 100 500 1,800
P($x$) : 0.50 0.40 0.10
E($x$) = Rs 430; 0.5 E($x$) = Rs 215

**7.5** P(A) = 0.2; P(B) = 0.25; P(C) = 0.4; and $P(\bar{A}) = 0.8$; $P(\bar{B}) = 0.75$; $P(\bar{C}) = 0.6$

$E(x) = x_1 P(\bar{A}) + x_2 P(\bar{B}) + x_3 P(\bar{C}) = 32$

**7.6** $P(\text{Win}) = \frac{9999}{10,000}$ and $\frac{1}{1000}$

$E(x) = -1 \times \frac{9999}{10,000} + 4999 \times \frac{1}{1000} = \text{Rs } 3.9991$

**7.7** Expected demand for transistors, $E(n) = \Sigma np = 3.62$
$E(C) = (10,000 + 500n) = 10,000 + 50\,E(n)$
= Rs 11,810.

**7.8** E($x$) = 508.

**7.9** EMV (A) = 6 × 0.4 + 7 × 0.3 + 8 × 0.2 + 9 × 0.1
= Rs 70,000.
EMV (B) = 6 × 0.1 + 7 × 0.2 + 8 × 0.5 + 9 × 0.2
= Rs 78,000.
Firm B will be more inclined towards the offer.

**7.10** E ($x$) = 2.75

**7.11**

| *State of Nature* | *Mild* | *Cold* | *Very Cold* |
|---|---|---|---|
| Prob. P($x$) | 0.40 | 0.32 | 0.28 |
| Sale of coat | 1000 | 1300 | 2000 |
| Profit, $x$ | 1000× (248 – 173) | 1300× (248 – 173) | 2000× (248 – 173) |

E(Profit) = Rs 1,03,200

## 7.5 DISCRETE PROBABILITY DISTRIBUTIONS

### 7.5.1 Binomial Probability Distribution

Binomial probability distribution is a widely used probability distribution for a discrete random variable. This distribution describes discrete data resulting from an experiment called a *Bernoulli process* (named after Jacob Bernoulli, 1654–1705, the first of the Bernoulli family of Swiss mathematicians). For each trial of an experiment, *there are only two possible complementary* (*mutually exclusive*) outcomes such as, defective or good, head or tail, zero or one, boy or girl. In such cases the outcome of interest is referred to as a '*success*' and the other as a '*failure*'. The term 'binomial' literally means two names.

**Bernoulli process:** A process in which each trial has only two possible outcomes, the probability of the outcome at any trial remains fixed over time, and the trials are statistically independent.

> ***Bernoulli process:*** *It is a process wherein an experiment is performed repeatedly, yielding either a success or a failure in each trial and where there is absolutely no pattern in the occurrence of successes and failures. That is, the occurrence of a success or a failure in a particular trial does not affect, and is not affected by, the outcomes in any previous or subsequent trials. The trials are independent.*

**Conditions for Binomial Experiment** The Bernoulli process involving a series of independent trials, is based on certain conditions as under:

(i) There are only two mutually exclusive and collective exhaustive outcomes of the random variable and one of them is referred to as a *success* and the other as a *failure*.
(ii) The random experiment is performed under the same conditions for a fixed and finite (also discrete) number of times, say $n$. Each observation of the random variable in an random experiment is called a *trial*. Each trial generates either a *success* denoted by $p$ or *a failure* denoted by $q$.
(iii) The outcome (i.e., success or failure) of any trial is not affected by the outcome of any other trial.
(iv) All the observations are assumed to be independent of each of each other. This means that the probability of outcomes remains constant throughout the process. Thus, the probability of a sucess, denoted by $p$, remains constant from trial to trial. The possability of a failure is $q = 1 - p$.

To understand the Bernoulli process, consider the coin tossing problem where 3 coins are tossed. Suppose we are interested to know the probability of two heads. The possible sequence of outcomes involving two heads can be obtained in the following three ways: HHT, HTH, THH.

The probability of each of the above sequences can be found by using the multiplication rule for independent events. Let the probability of a head be $p$ and the probability of tail be $q$. The probability of each sequence can be written as:

$$ppq \quad pqp \quad qpp$$

Each of these probabilities can be written as $p^2q$, they are all equal.

Since three sequences correspond to the same event '2 heads', therefore the probability of 2 heads in 3 tosses is obtained by using the addition rule of probabilities for mutually exclusive events. Since the probability of each sequence is same, we can multiply $p^2q$ (probability of one sequence) by 3 (number of possible sequences or orderings of 2 heads). Hence

$$P(2 \text{ heads}) = 3p^2q = {}^3C_2 p^2q$$

Here it may be noted that the possible sequences equals the binomial coefficient ${}^3C_2 = 3$. This coefficient represents the number of ways that three symbols, of which two are alike (i.e., 2H and one T), can be ordered (or arranged). In general, the binomial coefficient ${}^nC_r$ represents the number of ways that $n$ symbols, of which $r$ are alike, can be ordered.

Since events H and T are equally likely and mutually exclusive, therefore $p = 0.5$ and $q = 0.5$ for a toss of the coin. Thus the probability of 2 heads in 3 tosses, is

$$P(x = 2 \text{ heads}) = {}^3C_2\,(0.5)^2\,(0.5) = 3\,(0.25)\,(0.5) = 0.375$$

**Binomial Probability Function** In general, for a binomial random variable, $x$ the probability of success (occurrence of desired outcome) $r$ number of times in $n$ independent trials, regardless of their order of occurrence is given by the formula:

$$P\,(x = r \text{ successes}) = {}^nC_r\, p^r q^{n-r} = \frac{n!}{r!\,(n-r)!}\, p^r\, q^{n-r},\ r = 0, 1, 2, ..., n \quad (7\text{-}1)$$

where $n$ = number of trials (specified in advance) or sample size
$p$ = probability of success
$q$ = $(1 - p)$, probability of failure
$x$ = discrete binomial random variable
$r$ = number of successes in $n$ trials

In formula (7-1), the term $p^r\, q^{n-r}$ represents the probability of one sequence where $r$ number of events (called successes) occur in $n$ trials in a particular sequence, while the term ${}^nC_r$ represents the number of possible sequences (combinations) of $r$ successes that are possible out of $n$ trials.

**Binomial distribution:** A discrete probability distribution of outcomes of an experiment known as a Bernoulli process.

**Characteristics of the Bionomial Distribution** The expression (7-1) is known as **binomial distribution** with parameters $n$ and $p$. Different values of $n$ and $p$ identify different binomial distributions which lead to different probabilities of $r$-values. The *mean* and *standard deviation* of a binomial distribution are computed in a shortcut manner as follows:

Mean, $\mu = np$,

Standard deviation, $\sigma = \sqrt{n\,p\,q}$

Knowing the values of first two central moments $\mu_0 = 1$ and $\mu_1 = 1$, other central moments are given by

Second moment, $\mu_2 = npq$
Third moment, $\mu_3 = npq\,(q - p)$
Fourth moment, $\mu_4 = 3n^2p^2q^2 + npq\,(1 - 6pq)$

so that $\gamma_1 = \sqrt{\beta_1} = \dfrac{\mu_3}{\mu_2^{3/2}} = \dfrac{q-p}{\sqrt{npq}}$, where $\beta_1 = \dfrac{n^2p^2q^2(q-p)^2}{n^3p^3q^3}$

and $\gamma_2 = \beta_2 - 3 = \dfrac{\mu_4}{\mu_2^2} - 3 = \dfrac{1-6pq}{npq}$, where $\beta_2 = \dfrac{3n^2p^2q^2 + npq(1-6pq)}{n^2p^2q^2}$

For a binomial distribution, *variance* < *mean*. This distribution is unimodal when $np$ is a whole number, and mean = mode = $np$.

A binomial distribution satisfies both the conditions of *pdf*, because

$$P(x = r) \geq 0 \text{ for all } r = 0, 1, 2, \ldots, n$$

$$\sum_{r=0}^{n} P(x=r) = \sum_{r=0}^{n} \left[{}^nC_r\, p^r\, q^{n-r}\right] = (p + q)^n = 1$$

**Plotting the Binomial Distributions** Let us examine graphically the characteristics of binomial distributions when its parameters $n$ and $p$ change.

(i) Figure 7.5 illustrates the general shape of a family of binomial distributions with constant $n = 5$ and $p$ varies from 0.3 to 0.7.

In the three cases shown in Fig. 7.5, the skewness varies with the value of $p$. When $p$ is small (i.e. $p < 0.5$), the distribution is skewed to the right. When $p$ and $q$ are equal (i.e. $p = q = 0.5$), the distribution is symmetric. When $p$ is large (i.e. $p > 0.5$), the distribution is skewed to left.

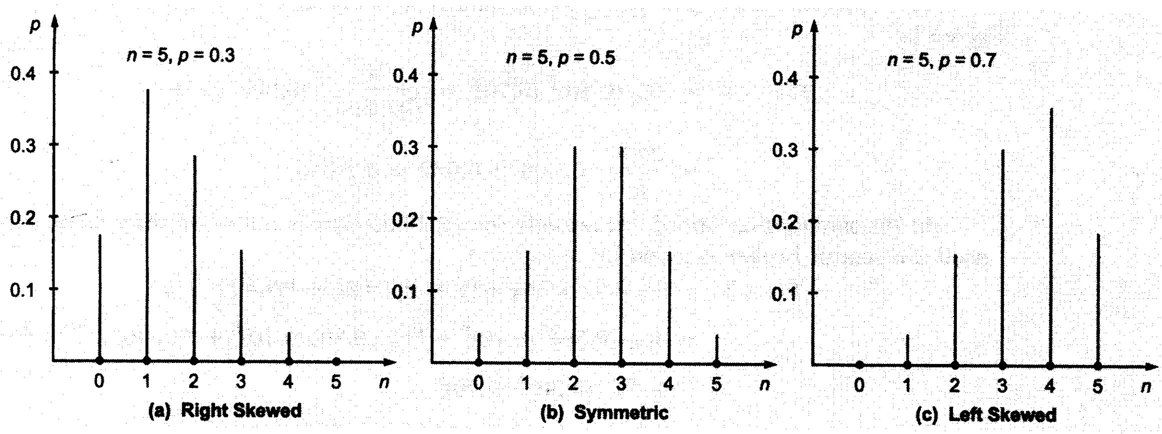



**Figure 7.5** Bionomial Distributions with Constants $n$ and Variable $p$

The probability of success is always in the vicinity of the mean, which always increases as $p$ increases. The variance is largest when $p$ and $q$ are equal. The smaller the variance, the larger the probability that a value of random variable, $x$ falls within the vicinity of the mean.

(ii) If $p$ stays constant but $n$ is increased, then for any value of $p$ other than 0.5 the binomial distribution approaches symmetry. In general, the smaller the value of $p$, the larger the sample size necessary for the symmetry to occur.

**Fitting a Binomial Distribution** A binomial distribution can be fitted to the observed values in the data set as follows:

(i) Find the value of $p$ and $q$. If one of these is known, the other can be obtained by using the relationship $p + q = 1$.

(ii) Expand $(p + q)^n = p^n + {}^nC_1 p^{n-1} q + {}^nC_2 p^{n-2} q^2 + \dots + {}^nC_r p^{n-r} q^r + \dots + {}^nC_n q^n$ using the concept of binomial theorem.

(iii) Multiply each term in the expansion by the total number of frequencies, N, to obtain the expected frequency for each of the random variable value.

The following recurrence relation can be used for fitting of a binomial distribution:

$$f(r) = {}^nC_r p^r q^{n-r}$$

$$f(r+1) = {}^nC_{r+1} p^{r+1} q^{n-r-1}$$

$$\therefore \quad \frac{f(r+1)}{f(r)} = \frac{p}{q}\frac{n-r}{r+1} \quad \text{or } f(r+1) = \frac{p}{q}\frac{n-r}{r+1} f(r)$$

For $r = 0$, $\quad f(1) = \frac{p}{q} n f(0)$

For $r = 1$, $\quad f(2) = \frac{p}{q}\frac{n-1}{2} f(1) = \left(\frac{p}{q}\right)^2 \frac{n(n-1)}{2!} f(0)$

For $r = 2$, $\quad f(3) = \frac{p}{q}\frac{n-2}{3} f(2) = \left(\frac{p}{q}\right)^3 \frac{n(n-1)(n-2)}{3!} f(0) \qquad (7\text{-}2)$

and so on.

In formula (7-2), we need to calculate $f(0)$, which is equal to $q^n$, where $q$ can be calculated from the given data.

**Example 7.8:** A brokerage survey reports that 30 per cent of individual investors have used a discount broker, i.e. one which does not charge the full commission. In a random sample of 9 individuals, what is the probability that

(a) exactly two of the sampled individuals have used a discount broker?

(b) not more than three have used a discount broker

(c) at least three of them have used a discount broker

**Solution:** The probability that individual investors have used a discount broker is, $p = 0.30$, and therefore $q = 1 - p = 0.70$

(a) Probability that exactly 2 of the 9 individual have used a discount broker is given by

$$P(x = 2) = {}^9C_2 (0.30)^2 (0.70)^7 = \frac{9!}{(9-2)!\,2!} (0.30)^2 (0.70)^7$$

$$= \frac{9\times 8}{2} \times 0.09 \times 0.082 = 0.2656$$

(b) Probability that out of 9 randomly selected individuals not more than three have used a discount broker is given by

$$P(x \le 3) = P(x=0) + P(x=1) + P(x=2) + P(x=3)$$

$$= {}^9C_0 (0.30)^0 (0.70)^9 + {}^9C_1 (0.30)(0.70)^8 + {}^9C_2 (0.30)^2 (0.70)^7 + {}^9C_3 (0.30)^3 (0.70)^6$$

$$= 0.040 + 9\times 0.30\times 0.058 + 36\times 0.09\times 0.082 + 84\times 0.027\times 0.118$$

$$= 0.040 + 0.157 + 0.266 + 0.268 = 0.731$$

(c) Probability that out of 9 randomly selected individuals at least three have a discount broker is given by

$$P(x \ge 3) = 1 - P(x < 3) = 1 - [P(x=0) + P(x=1) + P(x=2)]$$

$$= 1 - [0.040 + 0.157 + 0.266] = 0.537$$

**Example 7.9:** Mr Gupta applies for a personal loan of Rs 1,50,000 from a nationalised bank to repair his house. The loan offer informed him that over the yeras bank has received about 2920 loan applications per year and that the probability of approval was, on average, above 0.85

(a) Mr Gupta wants to know the average and standard deviation of the number of loans approved per year.

(b) Suppose bank actually received 2654 loan applications per year with an approval probability of 0.82. What are the mean and standard deviation now?

**Solution:** (a) Assuming that approvals are independent from loan to loan, and that all loans have the same 0.85 probability of approval. Then

Mean, $\mu = np = 2920 \times 0.85 = 2482$

Standard deviation, $\sigma = \sqrt{npq} = \sqrt{2920\times 0.85\times 0.15} = 19.295$

(b) Mean, $\mu = np = 2654 \times 0.82 = 2176.28$

Standard deviation, $\sigma = \sqrt{npq} = \sqrt{2654\times 0.82\times 0.18} = 19.792$

**Example 7.10:** Suppose 10 per cent of new scooters will require warranty service within the first month of its sale. A scooter manufacturing company sells 1000 scooters in a month,

(a) Find the mean and standard deviation of scooters that require warranty service

(b) Calculate the moment coefficient of skewness and kurtosis of the distribution.

**Solution:** Given that $p = 0.10$, $q = 1 - p = 0.90$ and $n = 1000$

(a) Mean, $\mu = np = 1000 \times 0.10 = 100$ scooters

Standard deviation, $\sigma = \sqrt{npq} = \sqrt{1000\times 0.10\times 0.90} = 10$ scootors (approx.)

(b) Moment coefficient of skewness

$$\gamma_1 = \sqrt{\beta_1} = \frac{q-p}{\sqrt{npq}} = \frac{0.90-0.10}{9.48} = 0.084$$

Since $\gamma_1$ is more than zero, the distribution is positively skewed.

Moment coefficient of kurtosis, $\gamma_2 = \beta_2 - 3$

$$= \frac{1-6pq}{npq} = \frac{1-6(0.10)(0.90)}{90} = \frac{0.46}{90} = 0.0051$$

Since $\gamma_2$ is positive, the distribution is platykurtic.

**Example 7.11:** The incidence of occupational disease in an industry is such that the workers have 20 per cent chance of suffering from it. What is the probability that out of six workers 4 or more will come in contact of the disease?

*[Lucknow Univ., MBA, 1998; Delhi Univ., MBA, 2002]*

**Solution:** The probability of a worker suffering from the disease is, $p = 20/100 = 1/5$. Therefore $q = 1 - p = 1 - (1/5) = 4/5$.

The probability of 4 or more, that is, 4, 5, or 6 coming in contact of the disease is given by

$$P(x \geq 4) = P(x = 4) + P(x = 5) + P(x = 6)$$

$$= {}^6C_4\left(\frac{1}{5}\right)^4\left(\frac{4}{5}\right)^2 + {}^6C_5\left(\frac{1}{5}\right)^5\left(\frac{4}{5}\right) + {}^6C_4\left(\frac{1}{5}\right)^6$$

$$= \frac{15\times16}{15625} + \frac{6\times4}{15625} + \frac{1}{15625} = \frac{1}{15625}(240 + 24 + 1)$$

$$= \frac{265}{15625} = 0.01695$$

Hence the probability that out of 6 workers 4 or more will come in contact of the disease is 0.01695.

**Example 7.12:** A multiple-choice test contains 8 questions with 3 answers to each question (of which only one is correct). A student answers each question by rolling a balanced dice and checking the first answer if he gets 1 or 2, the second answer if he gets 3 or 4, and the third answer if he gets 5 or 6. To get a distinction, the student must secure at least 75 per cent correct answers. If there is no negative marking, what is the probability that the student secures a distinction?

**Solution:** Probability of a correct answer, $p$ is one in three so that $p = 1/3$ and probability of wrong answer $q = 2/3$.

The required probability of securing a distinction (i.e., of getting the correct answer of at least 6 of the 8 questions) is given by:

$$P(x \geq 6) = P(x = 6) + P(x = 7) + P(x = 8)$$

$$= {}^8C_6\left(\frac{1}{3}\right)^6\left(\frac{2}{3}\right)^2 + {}^8C_7\left(\frac{1}{3}\right)^7\left(\frac{2}{3}\right) + {}^8C_8\left(\frac{1}{3}\right)^8$$

$$= \left(\frac{1}{3}\right)^6\left[{}^8C_6\left(\frac{2}{3}\right)^2 + {}^8C_7\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + {}^8C_8\left(\frac{1}{3}\right)^2\right]$$

$$= \frac{1}{729}\left[28\times\frac{4}{9} + 8\times\frac{2}{9} + \frac{1}{9}\right] = \frac{1}{729}(12.45 + 0.178 + 0.12)$$

$$= 0.0196$$

**Example 7.13:** The screws produced by a certain machine were checked by examining the number of defectives in a sample of 12. The following table shows the distribution of 128 samples according to the number of defective items they contained:

| No. of defectives in a sample of 12 : | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | |
|---|---|---|---|---|---|---|---|---|---|
| No. of samples : | 7 | 6 | 19 | 35 | 30 | 23 | 7 | 1 | = 128 |

(a) Fit a binomial distribution and find the expected frequencies if the chance of machine being defective is 0.5.

(b) Find the mean and standard deviation of the fitted distribution.

[*Delhi Univ., MBA, 2003*]

**Solution:** (a) The probability of a defective screw is, $p = 1/2$ and therefore $q = 1 - q = 1/2$; N = 128.

Since there are 8 terms, therefore $n = 7$. Thus, the probability that the defective items are 0, 1, 2, . . ., 7 is given by:

$$(p + q)^n = p^n + {}^nC_1\, p^{n-1} q + {}^nC_2\, p^{n-2} q^2 + \ldots + {}^nC_7\, p^{n-7} q^7$$

or

$$\left(\frac{1}{2}+\frac{1}{2}\right)^7 = \left(\frac{1}{2}\right)^7 + {}^7C_1\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right) + {}^7C_2\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^2 + \ldots + {}^7C_7\left(\frac{1}{2}\right)^7$$

$$= \left(\frac{1}{2}\right)^7 [1 + 7 + 21 + 35 + 35 + 21 + 7 + 1]$$

For obtaining the expected frequencies, multiply each term by N = 128. That is,

$$128\left(\frac{1}{2}+\frac{1}{2}\right)^7 = 128 \times \frac{1}{128}(1 + 7 + 21 + 35 + 35 + 21 + 7 + 1)$$

Thus, the expected frequencies are

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x$ : | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $f$ : | 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 |

(b) The mean of binomial distribution is given by $np$ and standard deviation by $\sqrt{npq}$. Given that, $n = 7, p = q = 1/2$. Thus

Mean= $np = 7\times(1/2) = 3.5$

Standard deviation $= \sqrt{npq} = \sqrt{7\times(1/2)\times(1/2)} = \sqrt{1.75} = 1.32$.

## Conceptual Questions 7B

**6.** (a) Define binomial distribution stating its parameters, mean, and standard deviation, and give two examples where such a distribution is ideally suited.

(b) Define binomial distribution. Point out its chief characteristics and uses. Under what conditions does it tend to Poisson distribution?

**7.** For a binomial distribution, is it true that the mean is the most likely value? Explain.

**8.** Demonstrate that the binomial coefficient ${}^nC_r$ equals ${}^nC_{n-r}$ and illustrate this with a specific numerical example.

**9.** What assumptions must be met for a binomial distribution to be applied to a real life situation?

**10.** What is meant by the term parameter of a probability distribution? Relate the concept to the binomial distribution?

**11.** What information is provided by the mean, standard deviation, and central moments of the binomial distribution?

**12.** What is a binomial coefficient and illustrate this with a specific numerical example.

## Self-Practice Problems 7B

**7.12** The normal rate of infection of a certain disease in animals is known to be 25 per cent. In an experiment with 6 animals injected with a new vaccine it was observed that none of the animals caught the infection. Calculate the probability of the observed result.

**7.13** Out of 320 families with 5 children each, what percentage would be expected to have (i) 2 boys and 3 girls, (ii) at least one boy? Assume equal probability for boys and girls.

**7.14** The incidence of a certain disease is such that on an average 20 per cent of workers suffer from it. If 10 workers are selected at random, find the probability that (*i*) exactly 2 workers suffer from the disease, (*ii*) not more than 2 workers suffer from the disease.

Calculate the probability upto fourth decimal place.

[*MD Univ., MCom, 1998*]

**7.15** The mean of a binomial distribution is 40 and standard deviation 6. Calculate $n, p$, and $q$.

[*Delhi Univ., MBA, 1998*]

**7.16** A student obtained answers with mean $\mu = 2.4$ and variance $\sigma^2 = 3.2$ for a certain problem given to him using binomial distribution. Comment on the result.

**7.17** The probability that an evening college student will graduate is 0.4. Determine the probability that out of 5 students (a) none, (b) one, and (c) at least one will graduate. [*Madras Univ., MCom, 1997*]

**7.18** The normal rate of infection of a certain disease in animals is known to be 25 per cent. In an experiment with 6 animals injected with a new vaccine it was observed that none of the animals caught infection. Calculate the probability of the observed result.

**7.19** Is there any inconsistency in the statement that the mean of a binomial distribution is 20 and its standard deviation is 4? If no inconsistency is found what shall be the values of $p$, $q$, and $n$.

**7.20** A multi-choice test consists of 8 questions with 3 answers to each question (of which only one is correct). A student answers each question by rolling a balanced dice and selects the first answer if he gets 1 or 2, the second if he gets 3 or 4, and the third answer if he gets 5 or 6. To get a distribution, the student must secure at least 75 per cent correct answers. If there is no negative, what is the probability that the student secures a distinction?

**7.21** Find the probability that in a family of 5 children there will be (i) at least one boy (ii) at least one boy and one girl (Assume that the probability of a female birth is 0.5).

**7.22** A famous advertising slogan claims that 4 out of 5 housewives cannot distinguish between two particular brands of butter. If this claim is valid and 5000 housewives are tested in groups of 5, how many of these groups will contain 0, 1, 2, 3, 4, and 5 housewives who do not distinguish between the two products? Assume that the capacity to distinguish between the two brands is randomly distributed so that Bernoulli trial conditions are satisfied.

**7.23** A supposed coffee connoisseur claims that he can distinguish between a cup of instant coffee and a cup of percolator coffee 75 per cent of the time. It is agreed that his claim will be accepted if he correctly identifies at least 5 out of 6 cups. Find (a) his chance of having the claim accepted if he is in fact only guessing, and (b) his chance of having the claim rejected when he does have the ability he claims.

## Hints and Answers

**7.12** Let P denote infection of the disease. Then $p = 25/100 = 1/4$ and $q = 3/4$.

$$P(x = 0) = {}^6C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^6 = \frac{729}{4096}$$

**7.13** (i) Given $p = q = 1/2$

$$P(\text{boy} = 2) = {}^5C_2\, p^2 q^3 = {}^5C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 = \frac{5}{16} \text{ or } 31.25\%$$

(ii) $$P(\text{boy} \geq 1) = 1 - {}^5C_0\, q^5 = 1 - \frac{1}{32} = \frac{31}{32} \text{ or 97 per cent.}$$

**7.14** Probability that a worker suffers from a disease, $p = 1/5$ and $q = 4/5$.

$$P(x = r) = {}^nC_r\, q^{n-r} p^r = {}^{10}C_r \left(\frac{1}{5}\right)^r \left(\frac{4}{5}\right)^{10-r} = {}^{10}C_r \frac{4^{10-r}}{5^{10}};\ r = 0, 1, 2, \ldots, 10$$

(i) $$P(x = 2) = {}^{10}C_2 \frac{4^{10-2}}{5^{10}} = 0.302$$

(ii) $$P(x = 0) + P(x = 1) + P(x = 2) = \frac{1}{5^{10}}\left({}^{10}C_0 4^{10} + {}^{10}C_1 4^9 + {}^{10}C_2 4^8\right) = 0.678.$$

**7.15** Given $\mu = np = 40$ and $\sigma = \sqrt{npq} = 6$. Squaring $\sigma$, we get $npq = 36$ or $40q = 36$ or $q = 0.9$. Then $p = 1 - q = 0.28$.
Since $np = 40$ or $n = 40/p = 40/0.1 = 400$.

**7.16** Given $\sigma^2 = npq = 3.2$ and $\mu = np = 2.4$. Then $2.4q = 3.2$ or $q = 3.2/2.4 = 1.33$ (inconsistent result)

**7.17** Given $p = 0.4$ and $q = 0.6$

(a) $P(x = \text{no graduate}) = {}^5C_0\, (0.4)\, (0.6)^5 = 1 \times 1 \times 0.0777 = 0.0777$

(b) $P(x = 1) = {}^5C_1\, (0.4)^1\, (0.6)^4 = 0.2592$

(c) $P(x \geq 1) = 1 - P(x = 0) = 1 - 0.0777 = 0.9223$

**7.18** Probability of infection of disease $= 25/100 = 0.25$; $q = 1 - p = 0.75$.

The first term in the expansion of $(q + p)^n = \left(\frac{3}{4} + \frac{1}{4}\right)^6$ is ${}^6C_0 \left(\frac{3}{4}\right)^6 = 0.177$, which is also the required probability.

**7.19** Given $\mu = np = 20$ and $\sigma = \sqrt{npq} = 4$ or $npq = 16$ or $20q = 16$ or $q = 16/20 = 0.80$ and then $p = 1 - q = 0.20$. Hence $npq = 16$ gives $n = 16/pq = 16/(0.20 \times 0.80) = 100$.

**7.20** Probability of correct answer, $p = 1/3$ and wrong answer, $q = 2/3$.

Probability of securing distinction (i.e., answering at least 6 of 8 questions correctly). That is,

$P(x \geq 6) = P(x = 6) + P(x = 7) + P(x = 8)$

$$= {}^8C_6 \left(\frac{1}{3}\right)^6 \left(\frac{2}{3}\right)^2 + {}^8C_7 \left(\frac{1}{3}\right)^7 \left(\frac{2}{3}\right) + {}^8C_8 \left(\frac{1}{3}\right)^8$$

$$= \left(\frac{1}{3}\right)^6 \left[28 \times \frac{4}{9} + 8 \times \frac{1}{3} \times \frac{2}{3} + \frac{1}{9}\right] = \frac{1}{729} \times \frac{129}{9}$$

$= 0.019$

**7.21** Since $p = q = 0.5$, therefore

(i) $P(\text{boy} = 0) = {}^5C_0\,(0.5)^0\,(0.5)^5 = 0.031$

$P(\text{at least one boy}) = 1 - 0.031 = 0.969$

(ii) $P(\text{at least 1B and 1G}) = {}^5C_1\,(0.5)^1\,(0.5)^4 + {}^5C_2\,(0.5)^2\,(0.5)^3 + {}^5C_3\,(0.5)^3\,(0.5)^2 + {}^5C_4\,(0.5)^4\,(0.5)$

$$= \frac{5}{32} + \frac{10}{32} + \frac{10}{32} + \frac{5}{32} = \frac{30}{32}$$

**7.22** $p$ = probability that 4 out of 5 cannot distinguish between two brands = 4/5 = 0.8

so that $q = 1 - p = 1/5 = 0.2$

Expected distribution containing 0, 1, . . ., 5 housewives who do not distinguish between two brands

$= {}^5C_0\,(0.2)^5 + {}^5C_1\,(0.8)\,(0.2)^4 + {}^5C_2\,(0.8)^2\,(0.2)^3 + {}^5C_3\,(0.8)^3\,(0.2)^2 + {}^5C_4\,(0.8)^4\,(0.2)^1 + {}^5C_5\,(0.8)^5$

Thus required number in each group would be

$5000\,(0.2)^5$; $5000\,{}^5C_1\,(0.8)\,(0.2)^4$; $5000\,{}^5C_2\,(0.8)^2\,(0.2)^3$; $5000\,{}^5C_3\,(0.8)^3\,(0.2)^2$; $5000\,{}^5C_4\,(0.8)^4\,(0.2)$; $5000\,(0.8)^5$

**7.23** Given $p$ = probability that he is capable of making a distinction = 0.75; $q = 1 - p = 0.25$

(i) $P(x < 5) = 1 - P(x \geq 5) = 1 - [{}^6C_5\,(0.75)^5\,(0.25) + {}^6C_6\,(0.75)^6]$

$= 1 - 0.534 = 0.466$

(ii) $P(x \geq 5) = 0.534$

### 7.5.2 Poisson Probability Distribution

Poisson distribution is named after the French mathematician S. Poisson (1781–1840), The Poisson process measures the number of occurrences of a particular outcome of a discrete random variable in a *predetermined time interval, space, or volume,* for which an *average number* of occurrences of the outcome is known or can be determined. In the Poisson process, the random variable values need counting. Such a count might be (i) number of telephone calls per hour coming into the switchboard, (ii) number of fatal traffic accidents per week in a city/state, (iii) number of patients arriving at a health centre every hour, (iv) number of organisms per unit volume of some fluid, (v) number of cars waiting for service in a workshop, (vi) number of flaws per unit length of some wire, and so on. The Poisson probability distribution provides a simple, easy-to compute and accurate approximation to a binomial distribution when the probability of success, $p$ is very small and $n$ is large, so that $\mu = np$ is small, preferably $np > 7$. It is often called the '*law of improbable*' events meaning that the probability, $p$, of a particular event's happening is very small. As mentioned above **Poisson distribution** occurs in business situations in which there are a few successes against a large number of failures or vice-versa (i.e. few successes in an interval) and has single independent events that are mutually exclusive. Because of this, the probability of success, $p$ is very small in relation to the number of trials $n$, so we consider only the probability of success.

**Poisson distribution:** A discrete probability distribution in which the probability of occurrence of an outcome within a very small time period is very small, and the probability that two or more such outcomes will occur within the same small time interval is negligible. The occurrence of an outcome within one time period is independent of the other.

**Conditions for Poisson Process** The use of Poisson distribution to compute the probability of the occurrence of an outcome during a specific time period is based on the following conditions:

(i) The outcomes within any interval occur randomly and independently of one another.

(ii) The probability of one occurrence in a small time interval is proportional to the length of the interval and independent of the specific time interval.

(iii) The probability of more than one occurrence in a small time interval is negligible when compared to the probability of just one occurrence in the same time interval.

(iv) The average number of occurrences is constant for all time intervals of the same size.

**Poisson Probability Function** If the probability, $p$ of occurrence of an outcome of interest (i.e., success) in each trial is very small, but the number of independent trials $n$ is sufficiently large, then the average number of times that an event occurs in a certain period of time or space, $\lambda = np$ is also small. Under these conditions the binomial probability function